

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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*Dynamics and Control of Flexible  
Space Vehicles*

*P. W. Likins*

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JET PROPULSION LABORATORY  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
PASADENA, CALIFORNIA

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## Preface

The work described in this report was performed under the cognizance of the Guidance and Control Division of the Jet Propulsion Laboratory.

The purpose of the report is to document and disseminate information of practical value to the engineer charged with the responsibility for developing an attitude-control system for a flexible space vehicle. It is essentially a treatise on the *dynamics* of flexible vehicles as viewed by the control systems engineer, and deals with control system design only to the extent of suggesting ways in which vehicle flexibility can be accommodated in both preliminary design and final evaluation of the attitude-control system.

The major portion of the report is concerned with the derivation of equations of motion for the computer simulation of a controlled and nonrigid vehicle, and the development of coordinate transformations that facilitate simulation. Three basic approaches to this simulation are covered: discrete-coordinate methods (Section II), hybrid-coordinate methods (Section III), and vehicle normal-coordinate methods (Section IV).

The discrete-coordinate methods of Section II involve few restrictions or approximations, and in some cases are as general as Newton's laws for the simulation of the dynamic response of a collection of interconnected rigid bodies. The limitations of these methods stem primarily from the difficulty of creating the required mathematical model of a real vehicle without exceeding the practical limits imposed on computation by considerations of budget, schedule, and computer capacity.

The hybrid-coordinate methods of Section III receive the greatest emphasis in this report, both because they appear to be most useful and because they are the least familiar. These methods may be applied only when some portions of the vehicle (flexible appendages) undergo deformations that may reasonably be assumed to remain "small," thereby permitting the transformation to modal coordinates for vehicle appendages. The key feature of this approach, as opposed to the discrete-coordinate method, is the possibility of truncating the matrix of modal coordinates.

The vehicle normal-coordinate methods of Section IV involve transformations of all the kinematic coordinates of the simulation, and not merely the appendage deformation coordinates. These methods are accordingly more limited, and even when applicable, they may require more complex coordinate transformations than the hybrid-coordinate methods would involve. In the simplest cases, however, the vehicle normal-coordinate methods probably afford the most efficient simulation, since they then permit the most severe coordinate truncation.

In addition to the material in this report for use in the simulation of a controlled flexible vehicle, there is material of value in preliminary control system design. Because of the modal coordinates employed in both the hybrid-coordinate methods and the vehicle normal-coordinate methods, the corresponding equations of motion permit simulations of varying dimension and complexity, depending on the degree of truncation. As shown in Section V, one can truncate so severely as to represent a very complex structural appendage by a single modal coordinate, and then manually calculate dynamic transfer functions for use in preliminary control system design.

## Acknowledgment

A work of this size inevitably reflects the cooperative effort of many people. Among those who actively supported the initiation of this undertaking were Mr. Boris Dobrotin and Mr. H. Karl Bouvier of the Spacecraft Control Section, and Mr. Ben Wada and Mr. Robert Oliver of the Applied Mechanics Section. The work was conducted under the cognizance of Mr. Bouvier.

The original work described in this report (the hybrid-coordinate method of dynamic simulation) was developed from a suggestion offered by Mr. Paul McHorney of Hughes Aircraft Company in the context of a particular problem. Mr. Austin Gale of Hughes Aircraft Company developed a computer simulation program that was useful in the early stages of development of this method.

The technical contributions of JPL personnel to this report are gratefully acknowledged. Of particular value were the critical evaluations of Dr. Marc Trubert, Mr. John Garba, and Mr. Robert Bamford, and the theoretical and computational contributions of Mr. Gerald Fleischer, Mr. Roy Mankovitz, and Mr. Ed Dorroh.

The editorial suggestions of Mrs. Vivia Crew are acknowledged with sincere appreciation.

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## **Abstract**

The purpose of this report is twofold: (1) to survey the established analytic procedures for the simulation of controlled flexible space vehicles, and (2) to develop in detail methods that employ a combination of discrete and distributed ("modal") coordinates, i.e., the hybrid-coordinate methods.

Analytic procedures are described in three categories: (1) discrete-coordinate methods, (2) hybrid-coordinate methods, and (3) vehicle normal-coordinate methods. Each of these approaches is described and analyzed for its advantages and disadvantages, and each is found to have an area of applicability.

The hybrid-coordinate method combines the efficiency of the vehicle normal-coordinate method with the versatility of the discrete-coordinate method, and appears to have the widest range of practical application.

The results in this report have practical utility in two areas: (1) complex digital computer simulation of flexible space vehicles of arbitrary configuration subject to realistic control laws, and (2) preliminary control system design based on transfer functions for linearized models of dynamics and control laws.



# Dynamics and Control of Flexible Space Vehicles

## I. Introduction

### A. Motivation

The development of an attitude-control system necessarily involves a dynamic simulation of the vehicle being controlled, but the accuracy required of that simulation may vary greatly from one application to another. As long as the attitude-control accuracy requirements are low, and the vehicle is relatively rigid, the "dynamics block" in a control system block diagram is easily generated. Modern space vehicles are far from rigid, however, and attitude-control accuracy requirements are increasingly stringent, particularly for optical observations from space vehicles. Further improvements in control system performance depend in many instances on improved simulation of the vehicle dynamics. The importance of this subject is reflected in the existence of the NASA Space Vehicle Design Criteria Monograph, *Effects of Structural Flexibility on Spacecraft Control Systems* (Ref. 1). This document includes relevant case histories and references, as well as design recommendations.

The incorporation of vehicle flexibility into control system design is a requirement neither new nor unique to space vehicle applications. Designers of control systems

for missiles and aircraft have long ago found it necessary to consider vehicle flexibility in control system development. To some degree, the concepts developed in this earlier context can be applied intact to space applications (Refs. 2-10), but in many respects space vehicles are unique, and new concepts must be developed to deal with them.

The environmental history of every artificial satellite or spacecraft is marked by a brief interval of vigorous acceleration and vibration during boost, followed by prolonged functioning in a quiescent mode of operation characterized by extremely small loads and accelerations. Some missions may require a second period of violent acceleration for all or part of the system, but still there are two distinct and radically different dynamic environments to be considered. The universal solution to the dilemma this poses for the structural designer has been the adoption of lightweight (and extremely flexible) deployable appendages. The resulting vehicle is relatively compact and rigid during the launch phase of its history, but after boost termination it emerges like a butterfly from its cocoon, extending antennas and booms and unfurling solar-cell arrays until the structure has undergone complete metamorphosis.

Although modern space vehicles can be found in a wide variety of configurations, certain dynamic features are sufficiently common to be described as characteristic of space vehicles, as opposed to missiles, aircraft, or land and sea vehicles. Most current space vehicles can be described as the combination of one or more essentially rigid bodies with one or more extremely flexible bodies. This natural separation of the structural subsystems of the vehicles into two distinct regimes is a consequence of the widespread use of deployable appendages, which are much less commonly found on surface or airborne vehicles.

The structural subsystems of a spacecraft or satellite are often required to undergo substantial relative motions during mission performance, while large antennas, solar-cell arrays, instrument packages, or propulsion devices change their relative orientation. With the exceptions of variable-sweep and rotary-wing aircraft, most surface and air vehicles have dynamically significant moving parts only as semirigid rotors in the form of wheels, propellers, and propulsion subsystems. A rigid, symmetric, fixed-axis rotor is easily incorporated into a dynamic simulation of a vehicle, but the presence of a fully articulated flexible antenna on a space vehicle necessitates major changes in the formulation of its equations of motion, particularly when the relative motion of vehicle and antenna is subject to closed-loop, nonlinear control.

A different class of problem is introduced by the use of discrete damping devices in spacecraft subsystem vibration isolation or passive spin-stabilization nutation attenuation.

All of these characteristic features of space vehicles present problems in dynamic simulation, and some of these problems are quite difficult to resolve by application of the methodology of aircraft and missile control system design. There must be a comprehensive examination of the question of dynamic simulation for attitude control of space vehicles, and new approaches must be developed for applications that are beyond the scope of the efficient utilization of traditional methods of analysis.

## B. Scope

The present study is concerned with the development of methods for the dynamic simulation of flexible spacecraft. The emphasis in this report is on analytic simulations, although it is recognized that analytic results must generally be confirmed or corrected by experimental measurements on models or prototype vehicles.

Analytic methods treated here are restricted in their applicability to vehicles admitting of idealization as combinations of rigid bodies, particles, continuous elastic bodies, and (in special cases) fluids. The equations of motion of a continuous mechanical system (with continuous spatial variation of mass and flexibility) are partial differential equations. It is assumed in this report that any such equations descriptive of linearly elastic solid subsystems have been subjected to a "modal coordinate" transformation (as defined later in this section) and the resulting coordinates have been truncated to permit representation of system deformation with a finite number of modal coordinates. Thus continuous linearly elastic solids are admissible only in the sense that they can be represented either as a collection of interconnected rigid bodies or by a finite number of modal deformation coordinates. Fluids are admissible under three conditions only: (1) the fluid in a given container can be idealized as rigid, (2) the fluid can be represented by a finite number of modal coordinates, or (3) all aspects of the fluid dynamics can be ignored, except possibly the influence of the fluid on energy dissipation. In consequence of the formal exclusion of continuous mechanical elements, the equations of motion are ordinary (and not partial) differential equations.

Three distinct approaches to the analysis of flexible vehicles are treated in this report: (1) the discrete-coordinate formulation, (2) the hybrid-coordinate formulation, and (3) the vehicle normal-coordinate formulation. These methods are considered in turn in Sections II, III, and IV, with emphasis on the second method. A brief, qualitative description of each of the three general approaches is given in the following paragraphs.

The vehicle normal-coordinate method is the traditional approach to the vibration analysis of elastic systems. This method is well documented in texts (Refs. 11-13), and space applications can be found in technical reports and journal papers (Refs. 2-10), so in this report it is described only insofar as this seems necessary for completeness of the study and for comparison with less familiar methods.

In brief, this method consists of formulating equations of motion, whenever possible, as a system of independent (uncoupled) scalar second-order differential equations. For limited motions of a restricted class of physical systems, systematic procedures can be generated for the determination of the transformation necessary to change from some arbitrarily selected coordinate system to the coordinate system corresponding to uncoupled scalar equations of motion. In general, the new uncoupled co-

ordinates (called *normal coordinates*) do not correspond individually to the translations and rotations of discrete points or rigid bodies of the vehicle; instead, each coordinate is associated with a motion in which the entire vehicle participates. Coordinates that correspond to motion of more than one particle or rigid body of the system are called *distributed* or *modal coordinates*, as opposed to *discrete coordinates*. When all the dynamic equations of a system are uncoupled (as in the vehicle normal-coordinate method), the modal coordinates are called *normal-mode coordinates* for that system. Because the equations of motion are uncoupled, the vehicle can undergo motion in which only one of the scalar coordinates of the system participates. In this mode of motion, all points of the vehicle oscillate at the same frequency (the normal-mode frequency) and the vehicle undergoes periodic deformations into the same deformed shape (the normal-mode shape). Each modal coordinate defines the amount of the response in the corresponding mode, suitably normalized. The independence of these normal-mode coordinates permits the independent calculation of their participation in the vehicle motion. This is the key feature of the normal-coordinate approach, because it permits the exercise of engineering judgment in determining which coordinates are so significant as to warrant retention, and which may be abandoned in coordinate truncation.

Even this classical approach to vibration analysis has not been very widely used in spacecraft control system design and evaluation, so aspects of the method peculiar to this area of application are emphasized in this report. The assumptions underlying the vehicle normal-coordinate approach, and the theoretical and practical limitations on its use, are brought into focus. Appreciation of the restrictions implicit in this approach is particularly important in space vehicle application, because the method breaks down when the system includes nonlinearities, rotors, discrete dampers, or articulated moving parts, and these are precisely the features just described as characteristic of space vehicles. Formulating equations of motion as first-order (state) equations eliminates some of these obstacles, but the practice of working with normal-mode coordinates for the entire vehicle is still quite restricted.

All of these difficulties can be accommodated by avoiding modal coordinates entirely, relying upon a complete set of equations of a collection of interconnected rigid bodies considered to be representative of the vehicle. Because each rigid body of the system model is discrete, and the coordinates employed in this approach are coordinates of position and/or attitude of the individual bodies,

this approach is sometimes called the *discrete-parameter* or *discrete-coordinate* method.

The discrete-parameter approach to space vehicle simulation has received great emphasis since 1965, primarily because of its generality. A growing body of literature on this subject is becoming available (Refs. 14-22), and digital computer programs based on these and other formulations are becoming commonplace tools of analysis.

The approach most frequently adopted (Refs. 14-21) involves direct application of the Newton-Euler equations of translation and rotation to various subsets of bodies in the assembly (often to each of the bodies individually). Alternatively, Lagrange's equations are sometimes applied (Ref. 22).

When application is restricted to a vehicle model composed of  $n$  rigid bodies interconnected at  $n - 1$  point contacts so as to form a "tree" structure topologically, the Newton-Euler equations allow dramatic simplification. These restrictive assumptions are illustrated in Fig. 1, which shows that adjacent rigid bodies share at least one common point, and no closed loops are formed by any string of rigid bodies. The 12 bodies and 11 points

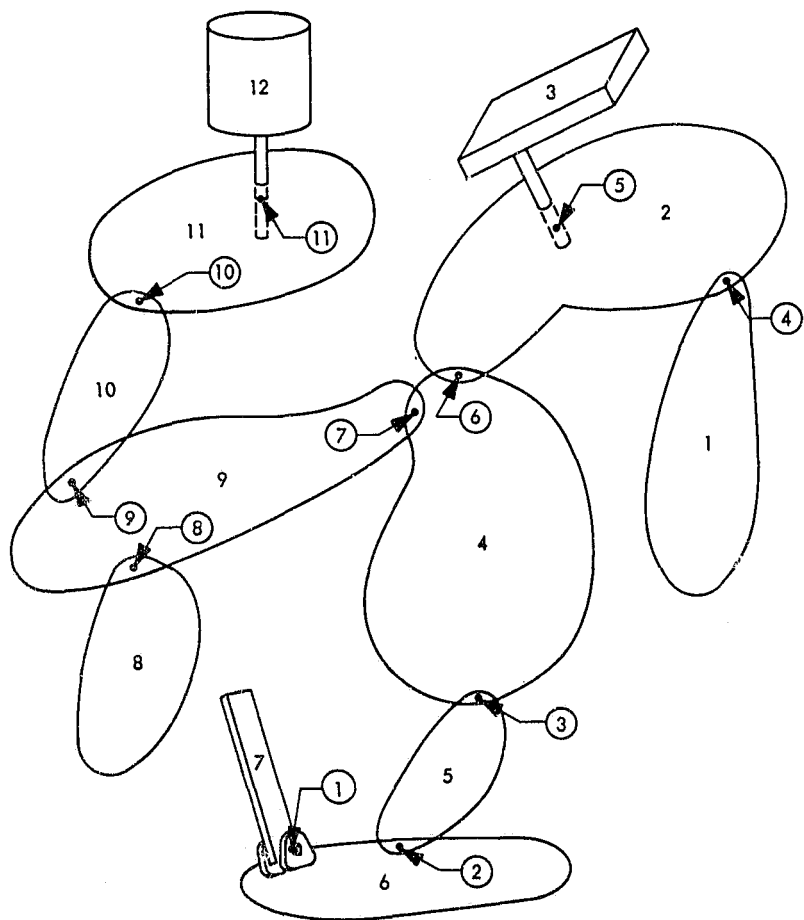


Fig. 1. Topological tree of point-connected rigid bodies

of contact in the figure are arbitrarily labeled. The simulations described in Refs. 14-19 and 21 are limited in this manner (the "tree" of Ref. 21 is more like a "bush," with  $\epsilon$ , trunk and limbs each consisting of a single rigid body). Under these conditions, for a complete kinematic description of the system, it is sufficient to know the location of one of its points (e.g., the mass center) and the orientation or attitude of each of the bodies. It is therefore possible to reduce the number of equations required for the dynamic simulation, eliminating all translational coordinates (except those of the system mass center, which may be determined independently in many cases).

Euler's equations of rotational motion of a rigid body (say, the  $i$ th body) are three scalar first-order differential equations in the variables  $\omega_1^i, \omega_2^i, \omega_3^i$ , which are measure numbers of the inertial angular velocity of that body for a body-fixed vector basis. These variables, collected for all  $n$  bodies, generally constitute the unknowns of the final differential equations, which are therefore first-order equations. Because a set of angular velocity measure numbers does not in itself provide a complete kinematic description of the system, an additional set of first-order equations must be included to permit the determination of attitude by angular velocity integration. These kinematic equations may be expressed in terms of any desired set of attitude parameters, e.g., direction cosines or a subset of direction cosines, Euler parameters, Euler-Rodrigues parameters (Ref. 23), Euler angles (3, 1, 3 rotations), or Tait-Bryan angles (1, 2, 3 rotations). The last three alternatives involve only three parameters (the minimum number) for the attitude description of each body, but all three-parameter sets are plagued by isolated singularities that make numerical computations impossible for certain attitudes (without coordinate transformation). Most spacecraft-simulation programs employ a larger set of attitude parameters, accepting the inconvenience of working with a set of coordinates interrelated by one or more constraint relationships. There appears to be no overwhelming preference between a subset of the direction cosines (usually six in number) and the four Euler parameters (or Cayley-Klein parameters or elements of a quaternion).

If one of the bodies of the set has more than one point in common with another body (e.g., the two are connected by a line hinge), this may be represented in the simulation by an additional constraint equation. Procedures exist for the systematic incorporation of such "joint constraints" into the dynamic equations, with consequent reduction in dimension of the matrix differential equa-

tion of the system. Analysts disagree, however, on the advisability of this operation for computational efficiency (Refs. 14, 18, and 20).

When Lagrange's equations are written in terms of a set of independent generalized coordinates (Ref. 24, article 26), constraint equations are absorbed automatically, and the equations of motion are obtained immediately as a set of second-order differential equations of minimum number. The restriction to the use of independent generalized coordinates may, however, be a handicap, since this precludes the adoption of Euler parameters or direction cosines for attitude description. As a result, such a conventional Lagrangian formulation must employ a coordinate system not entirely free of singularities. In practice, most (perhaps all) programs based on Lagrange's equations use attitude angles as coordinates (rather than Euler-Rodrigues parameters), and this selection introduces computationally cumbersome trigonometric functions into the calculations. This may be a greater handicap than the presence of singularities, particularly if digital simulation is anticipated. This approach is also restricted to systems with holonomic constraints, although this is perhaps not as severe a restriction in aerospace applications as it would be for land vehicle applications that include rolling wheels.

One may increase the scope and flexibility of the Lagrangian approach in at least two significant ways. With the introduction of Lagrange multipliers (Ref. 24, article 87), the restriction to a set of independent generalized coordinates is relaxed, and one may adopt any kinematically complete set of coordinates, providing only that constraint equations in the form of equalities (not inequalities) exist in sufficient number to offset the coordinate redundancy. (Unfortunately, aerospace vehicle subsystem constraints such as damper and gimbal "stops" are in the excluded class of inequalities.) This approach extends the scope to certain nonholonomic systems, and it permits the use of redundant attitude parameters (e.g., Euler parameters). The price paid is the added dimension of the problem. This method does not seem promising in comparison with the Newton-Euler method, and appears not to have been pursued for general space vehicle simulation. However, applications to restricted problems can be found in the aerospace journal literature (Ref. 25).

The second relevant modification of the Lagrangian formulation is more fundamental. It is possible to avoid the reliance in Lagrange's equations upon generalized coordinates, which must be sufficient to establish fully in

themselves the configuration of the system at any given time. One may alternatively write Lagrange's equations in terms of *quasi-coordinates* (Ref. 24, article 30), which are quantities whose differentials may be written as linear combinations of differentials of generalized coordinates and time. Since the angular velocity measure numbers  $\omega_1^i, \omega_2^i, \omega_3^i$  of a given  $i$ th body qualify as derivatives of quasi-coordinates, this Lagrangian approach can produce results with some of the qualities of the Newton-Euler formulation.

A recent adaptation by Kane and Wang (Refs. 26, 27) of the quasi-coordinate formulation seems well suited to complex spacecraft simulation. The method is applicable to certain nonholonomic systems and to systems with redundant (and constrained) coordinates; yet it automatically eliminates nonworking constraint forces and torques. Although the approach by Kane and Wang appears to combine certain advantages of the Newton-Euler method and the Lagrangian method, this path to complex system simulation has not yet been taken to the point of a multipurpose computer program for machine computation of the response of arbitrary discrete parameter systems.

Section II contains an introduction to the formulation of the equations of motion for discrete-parameter systems. The text describes without derivation the results of Refs. 14 and 15, and notes some of the features of Refs. 16-20. The method of Kane and Wang is also briefly described. In view of the functional and structural beauty of the equations developed in these references, it seems unlikely that improvement can be realized by additional independent derivations, and the analyst confronting the problem of simulating complex discrete-parameter systems is advised to acquaint himself first with the referenced literature.

Discrete-parameter simulations are not without serious disadvantages. Satisfactory simulation of real vehicles may require a great many rigid bodies in the model. The resulting differential equations are then of high dimension, and their digital solution may be plagued by high-frequency responses that are of no interest to the engineer. There is no mechanism for truncating the matrix of coordinates retained in the determination of vehicle response, so as to eliminate these high-frequency responses and reduce the dimension of the problem.

When the system configuration demands the generality of a discrete-parameter simulation, this alternative must

be adopted. But when all or part of the vehicle allows the assumption of small, linearly elastic deformation, the most efficient simulation is that which combines discrete coordinates with distributed (modal) coordinates, retaining the generality of discrete coordinates where necessary, but securing the computational advantages of (truncated) modal coordinates where possible. The result is a *hybrid-coordinate* system that permits accurate simulation of complex modern space vehicles with a minimum number of coordinates, and with irrelevant high-frequency oscillations removed from the integration output by modal-coordinate truncation. Although this method is a natural combination of the other two approaches, it has received very little emphasis in the technical literature to date (Refs. 28, 29). For this reason, and because this method seems to combine the computational advantages of modal analysis with the needed generality of the discrete-coordinate approach, the hybrid-coordinate method receives major emphasis in this report.

The question of control system simulation per se is included in this report only insofar as this issue is coupled to the primary question of flexible vehicle simulation.

## II. Discrete-Coordinate Methods

### A. Augmented-Body Methods

Hooker and Margulies (Ref. 14) and Roberson and Wittenberg (Refs. 15, 16) have observed that when a system of  $n$  point-connected rigid bodies is assembled in a topological tree (as in Fig. 1), certain inertia-like terms naturally appear in combination in the individual equations of motion of each of the rigid bodies in the set. These combinations admit of physical interpretation as the inertia dyadics (or tensors or matrices) of abstractions called the *augmented bodies*. Briefly, the  $i$ th augmented body consists of the  $i$ th body of the set together with certain particles (point masses) attached to each of the joints of that body. The point mass attached to a given joint of the  $i$ th body equals the total mass of all of the connected bodies located "outboard" of the joint. For example, the augmented body 9 of the 12-body system shown in Fig. 1 is illustrated in Fig. 2. The mass center of the augmented body is called the *connection barycenter* (or simply the barycenter). The inertia dyadic of the augmented body with respect to the corresponding barycenter is the term that appears in the equations.

A variety of approaches might be considered in formulating the Newton-Euler equations of motion of a system of rigid bodies such as illustrated in Fig. 1. Perhaps the most direct approach is to isolate each of the bodies in

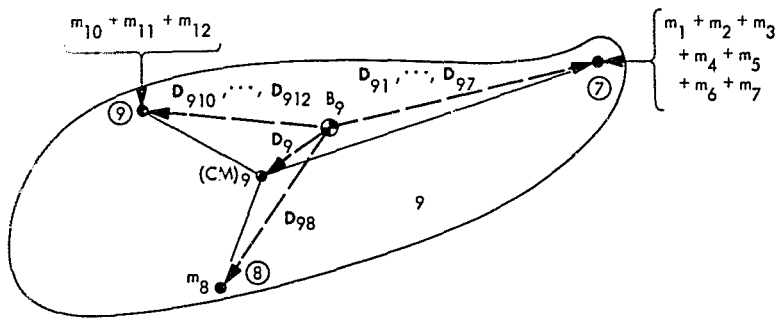


Fig. 2. Augmented body 9 and its barycenter  $B_9$

turn and write the translational and rotational vector equations for that body, recognizing that neighboring bodies contribute to the applied force and torque. This procedure leads to the appearance of augmented-body inertia dyadics, so it is the method used here. It should be recognized, however, that this is not the only possible approach; one might, for example, exclude the equations of motion of the  $n$ th body and instead write the translational and rotational equations of motion of the total system, or of any connected group of bodies within the system. Other possibilities are explored in Section II-B.

For an individual body, say, the  $i$ th body, the translational and rotational vector equations of motion may be written as

$$\left. \begin{aligned} \mathbf{F}^i &= m_i \mathbf{a}^i \\ \mathbf{T}^i &= \dot{\mathbf{H}}^i = \mathbf{I}^i \cdot \dot{\boldsymbol{\omega}}^i + \boldsymbol{\omega}^i \times \mathbf{I}^i \cdot \boldsymbol{\omega}^i \end{aligned} \right\} \quad (1)$$

where  $\mathbf{F}^i$  is the resultant force and  $\mathbf{T}^i$  the resultant torque applied to the  $i$ th body,  $m_i$  is the body mass,  $\mathbf{a}^i$  the mass center inertial acceleration,  $\mathbf{H}^i$  the body angular momentum,  $\mathbf{I}^i$  the inertia dyadic, and  $\boldsymbol{\omega}^i$  the inertial angular velocity. The quantities  $\mathbf{T}^i$ ,  $\mathbf{H}^i$ , and  $\mathbf{I}^i$  are all referred to the body mass center. In the notation used in this report, dot ( $\dot{\phantom{x}}$ ) over a vector indicates time differentiation in an inertial reference frame.

Among the constituents of the applied force  $\mathbf{F}^i$  there are "hinge forces," i.e., forces applied to the  $i$ th body by contiguous bodies at the points of contact. These hinge forces generally contribute also to the torque  $\mathbf{T}^i$  about the  $i$ th body mass center, and typically this is the only coupling mechanism between the translational and rotational equations (1).

It is evident by inspection of Fig. 1 that knowledge of the attitude of every body of a point-connected set is sufficient for the complete determination of the system

configuration and attitude; it is not necessary to know mass-center position coordinates as well, since these must follow from the kinematic constraints imposed by the joints. Consequently it must be possible to combine the vector equations (1) for all  $n$  bodies in such a way as to obtain  $n$  vector equations of rotational motion that are internally complete, without any coupling terms to additional translational equations. This is accomplished by first solving the translational equations (1) for the hinge forces in terms of mass-center accelerations, and then obtaining from kinematics the mass-center accelerations in terms of geometry and rotational coordinates.

The specific manipulations that accomplish this reduction of equations (1) to a dynamically complete set of half as many equations can be found in Refs. 14–18. In the work of Hooker and Margulies (Ref. 14), the manipulation is performed at the level of explicit vector equations for the individual bodies, while in the work of Roberson and Wittenberg (Refs. 15, 16), this task is deferred to a stage of the derivation in which the rotational equations of motion are matrix equations of dimension  $3n$ . In both developments, there emerge terms (dyadics in Ref. 14 and matrices in Refs. 15, 16) that have the indicated interpretation as inertia dyadics or matrices of augmented bodies. The vector-dyadic result, for example, is given for the  $\lambda$ th body in the notation of Ref. 14 by

$$\begin{aligned} \Phi_\lambda^* \cdot \dot{\boldsymbol{\omega}}_\lambda + \boldsymbol{\omega}_\lambda \times \Phi_\lambda^* \cdot \boldsymbol{\omega}_\lambda = \\ \mathbf{T}_\lambda + \sum_{j \in J_\lambda} \mathbf{T}_{\lambda j}'' + \mathbf{D}_\lambda \times \mathbf{F}_\lambda + \sum_{\mu \neq \lambda} \mathbf{D}_{\lambda \mu} \times \mathbf{F}_\mu \\ + m_\lambda \sum_{\mu \neq \lambda} \mathbf{D}_{\lambda \mu} \times [\dot{\boldsymbol{\omega}}_\mu \times \mathbf{D}_{\mu \lambda} + \boldsymbol{\omega}_\mu \times (\boldsymbol{\omega}_\mu \times \mathbf{D}_{\mu \lambda})] \end{aligned} \quad (2)$$

with the following symbol definitions:

$\Phi_\lambda^*$  = the inertia dyadic of the  $\lambda$ th augmented body, referred to the corresponding barycenter

$\boldsymbol{\omega}_\lambda$  and  $\boldsymbol{\omega}_\mu$  = inertial angular velocities of bodies  $\lambda$  and  $\mu$ , respectively

$\mathbf{T}_\lambda$  = that portion of the resultant torque applied to body  $\lambda$  obtained by excluding forces and torques applied at joints

$\mathbf{T}_{\lambda j}''$  = the "hinge torque" applied at joint  $j$  of body  $\lambda$

$J_\lambda$  = the set of numeric labels for the joints on body  $\lambda$  (e.g., from Figs. 1 and 2,  $J_9$  includes 7, 8, and 9)

$\mathbf{D}_\lambda$  = the position vector from the barycenter  $B_\lambda$  to the mass center of body  $\lambda$  (e.g., see Fig. 2)

$\mathbf{F}_\lambda$  and  $\mathbf{F}_\mu$  = the forces applied to bodies  $\lambda$  and  $\mu$ , respectively, excluding forces applied at joints

$\mathbf{D}_{\lambda\mu}$  = the position vector from the barycenter  $B_\lambda$  to the joint of body  $\lambda$  that leads to body  $\mu$  (even if body  $\mu$  is not directly connected to body  $\lambda$ , but instead is part of a chain of bodies connected to body  $\lambda$ )

$\mathcal{M}$  = the total system mass

$\sum_{j \in J_\lambda}$  = the sum over values of  $j$  in the set  $J_\lambda$ .

The left side of Eq. (2) has exactly the form of the vector-dyadic representation of Euler's equations, except for the substitution of the augmented-body inertia-dyadic about the barycenter ( $\Phi_\lambda^*$ ) for the body inertia-dyadic about the mass center ( $\mathbf{I}^\lambda$ ). The right side of Eq. (2) includes the body torque  $\mathbf{T}_\lambda$  and the relevant hinge torques  $\mathbf{T}_{\lambda j}''$ , as would be the case for Euler's equations, but the torque contribution of hinge forces takes a surprising form. In every case, torques are calculated with respect to the barycenter, and the appropriate force turns out to be the sum of the body forces  $\mathbf{F}_\mu$  and certain "inertial forces" that may be attributed to angular accelerations and centripetal accelerations.

In view of the complexity of the  $n$ -body system under examination, it is astonishing that the equations of motion should be so simple in structure and amenable to physical interpretation. Simplification is even more dramatic when there is a coincidence of barycenter  $B_\lambda$  and a joint, since then the vector  $\mathbf{D}_{\lambda\mu}$  is zero for all bodies connected to body  $\lambda$  by that joint, and substantial decoupling of the equations results.

In general, of course, there are required  $n$  vector-dyadic equations such as Eq. (2) to complete the dynamic description, and in addition some specification of the hinge torques  $\mathbf{T}_{\lambda j}''$  is required. In most applications, some of the bodies in the model are connected by line hinges, so that a line is common to two bodies of the system. The hinge torques  $\mathbf{T}_{\lambda j}''$  transverse to the hinge axis then become constraint torques, and each line hinge provides two scalar constraint equations that preclude relative rotations except about the hinge axis. Similarly, a two-gimbal joint provides one constraint equation. If there

are  $\nu$  constraint equations and  $n$  rigid bodies in the system, then there are available  $3n + \nu$  scalar equations to be solved for the  $3n$  unknown angular velocity measure numbers plus the  $\nu$  constraint torques. (An additional equation is of course required for the determination of the translation of the system mass center, but this equation is generally uncoupled with the attitude equations and is ignored here.)

The most direct approach to the determination of dynamic response is to combine all the equations of dynamics and kinematics with the constraint equations as a single-matrix first-order differential equation of dimension  $6n + \nu$  and proceed with numerical integration.

The possibility of eliminating the  $\nu$  constraint torques, thus reducing the dimension of the problem to  $6n$ , is discussed briefly in Ref. 14, and related questions are explored extensively in Refs. 16, 18, 19, 20, 26, and 27. Some of the methods that serve this purpose are discussed in Sections II-B and II-C.

The task of assembling the  $n$  vector-dyadic rotational equations into a single matrix equation is made quite awkward by the summations over limited sets of joints and bodies that appear in Eq. (2). Roberson and Wittenberg (Refs. 15, 16), in a derivation that developed in parallel to that of Hooker and Margulies (Ref. 14), utilize graph theory to construct an elegant formalism for the systematic assemblage of the individual vector-dyadic equations into the single  $3n$ -dimensional matrix equation of motion of the total system. The equations of Ref. 15 are programmed (with some modification) in Ref. 17, and those of Ref. 14 provide the basis for the program developed in Ref. 18.

Whether the matrix equation is developed with the beautiful formalism devised by Roberson and Wittenberg or assembled in more pedestrian fashion (as computational efficiency may suggest), the final structure of the equation must be as follows:

$$\mathcal{A} \dot{\omega} = \mathcal{B} + \mathcal{L} \quad (3)$$

where  $\mathcal{L}$  absorbs all terms involving constraint torques,  $\mathcal{B}$  embraces all external forces and torques as well as all terms involving inertial angular velocity terms  $\omega_\alpha^i$  ( $\alpha = 1, 2, 3, i = 1, \dots, n$ ), and  $\mathcal{A}$  is the coefficient matrix of all differentiated terms, which are assembled in the column matrix  $\dot{\omega}$ . Ignoring for the moment the question of how constraint equations are to be used to accommodate unknowns in  $\mathcal{L}$ , one can readily see that



it is the nature of  $\mathcal{A}$  that will determine computational difficulty. If  $\mathcal{A}$  were a nonsingular diagonal matrix, a trivial inversion would permit Eq. (3) to be written as

$$\dot{\omega} = \mathcal{A}^{-1} \mathcal{B} + \mathcal{A}^{-1} \mathcal{L} \quad (4)$$

where superscript  $-1$  denotes matrix inverse.

Unfortunately, a glance at Eq. (2) indicates that inertial coupling terms involving  $\dot{\omega}_\mu$  must exist unless  $\mathbf{D}_{\lambda\mu} = 0$  for all  $\mu$ . The same term  $\mathbf{D}_{\lambda\mu} \times \dot{\omega}_\mu$  also assures the time-varying character of  $\mathcal{A}$ , since the cross product depends on the changing relative attitude of bodies  $\lambda$  and  $\mu$ . This constitutes a major obstacle to numerical integration, since if Eq. (4) is used it becomes necessary to invert the  $3n$  by  $3n$  matrix  $\mathcal{A}$  at each step of the integration. In practice, it may prove more convenient to retain the equation in the form of Eq. (3), applying Gaussian elimination rather than matrix inversion at each integration step. Even with this expedient, the algebraic process of finding  $\dot{\omega}$  is apt to consume most of the computer time in numerical integration. Since time required for this process increases roughly with the third power of the dimension of the matrix  $\mathcal{A}$ , abundant motivation exists for working with equations of minimum dimension.

## B. Nested-Body Methods

In writing  $n$  vector equations of rotational motion for a set of  $n$  point-connected rigid bodies, one must make a choice of the  $n$  material subsystems to be isolated for dynamic analysis; it is not obviously advantageous to isolate each of the rigid bodies individually, as is generally the practice in Refs. 14–18. (In Refs. 15–18 there is an explicit departure from this practice when a pair of rigid bodies consists of a rigid body containing a rigid, axisymmetric rotor on a fixed axis.)

Velman (Ref. 20) and Russell (Ref. 19) elect instead to write vector equations of motion in turn for  $n$  different subsets of bodies, including a final set of rotational and translational equations for the composite vehicle. For example, if the vehicle were modeled by 12 point-connected rigid bodies as shown in Fig. 1, both Velman and Russell might write equations of motion first for body 12, then for the subset including bodies 11 and 12, and then for 10, 11, 12. Next, body 8 might be isolated and its equations written, and these followed by equations for the subset including bodies 8–12. One might then begin anew at the end of another chain of bodies (either body 1 or body 7), and progress inward as far as possible without ever including more than one addi-

tional body in any one step, and without ever considering a subset with more than one joint connecting it with the excluded bodies of the system. For this example, one might follow the indicated pattern until each of the three chains attached to body 4 has been considered, without ever including body 4 itself. (This choice of body 4 is arbitrary, since other subset selections within the prescribed pattern can converge as well on any other body of the system.) Finally, the equations of motion of the composite vehicle are recorded, to complete the dynamic simulation.

The idea of isolating in sequence such subsets of rigid bodies in an  $n$ -body system (called *nested bodies* by Velman) seems to have both advantages and disadvantages. The concepts of the *augmented body* and the *connection barycenter* are helpful aids to physical interpretation (if not to numerical computation), and the terms with which these phrases are identified in the equations of Roberson-Wittenberg (Refs. 15, 16) and Hooker-Margulies (Ref. 14) do not appear when the nested-body approach is taken. In compensation, however, it would appear that the nested-body derivation facilitates the elimination of internal constraint forces and torques.

Velman (Ref. 20) derives his equations in a manner that leads quite naturally to the use of attitude variables that establish the orientation of each body of the system relative to an adjacent body (except of course for one reference body with inertial attitude parameters). Because geometrical constraints generally restrict relative motions, it is perhaps easier to identify which attitude variables are constrained when relative motion coordinates are employed. Nonetheless, there is no attempt to eliminate kinematically constrained variables in the derivation of Velman's dynamical equations, so his equations have at the outset the same basic structure and dimension as those of Roberson and Wittenberg or Hooker and Margulies, namely,

$$\mathcal{A} \dot{\omega} = \mathcal{B} + \mathcal{L} \quad (5)$$

These symbols differ in interpretation among the several authors, and, in particular, Velman's  $\omega$  includes some relative velocities. It should be noted also that Velman's simulation specifically includes a linearly oscillating particle, so the method is not restricted to a point-connected set of rigid bodies. In Eq. (5), then, one of the scalars in the matrix  $\omega$  is the translational velocity of a particle relative to its housing body. As in Refs. 14–16, the dimension of matrix  $\dot{\omega}$  is  $3n$  for an  $n$ -body system as long as each body



is either a particle or a rigid body forming part of a point-connected set. Again,  $\mathcal{A}$  and  $\mathcal{B}$  are matrices that depend upon the kinematic unknowns of the problem, and  $\mathcal{L}$  is the matrix of unknown constraint torques (and constraint forces for the oscillating particle). The most significant feature of Velman's work is the procedure he devises for the elimination from these equations of the unknown constraint forces and torques.

Velman notes that the effect of the constraints is usually to confine the solution of Eq. (5) to some linear manifold in the  $3n$ -dimensional space of  $\omega$ . If, for example, a line hinge connects two rigid bodies of the system, and three of the scalars in  $\omega$  are the relative rotation rates of these bodies about the hinge axis and two transverse axes, the effect of the constraints is to confine the solution to that subspace of the  $\omega$  space excluding the two transverse-axis rotation rates, which are constrained by the hinge to be zero. The subspace to which the solution is confined is then of dimension  $3n - 2$ . This fact is unchanged even when the choice of variables in  $\omega$  does not specifically include these three relative rates about and transverse to the hinge; this choice may simplify the explicit specification of the solution subspace, but it introduces no conceptual change in the argument. Similarly, the line of argument remains intact when the solution in a subspace of  $\omega$  is constrained to be not zero, but a specified function of time. If, for the previous example, the two bodies connected by a line hinge undergo a prescribed relative rotation about that hinge (as for an idealized scanning antenna), then the solution is a prescribed function of time in a three-dimensional subspace of the  $\omega$  space.

The preceding remarks amount to the observation that the constraints determine the projection of the motion on a particular linear manifold in the  $\omega$  space. Velman introduces the perpendicular projection operator  $\mathcal{P}$  (a  $3n$  by  $3n$  matrix called  $E$  in Ref. 20) and the analytic expression

$$\mathcal{P}\dot{\omega} = \mu(t) \quad (6)$$

to represent the specified motion  $\mu(t)$ , which is the constrained partial solution. Because the variables in  $\omega$  are generally the relative motions, the matrix  $\mathcal{P}$  is typically a diagonal matrix, null except for ones on the diagonal corresponding to constrained coordinates. For a two-body system consisting of a primary body with inertial rates  $\omega_1, \omega_2, \omega_3$  and a second body attached on a line hinge with prescribed relative rotation rates  $\omega_4 = 0, \omega_5 = 0, \omega_6 = \dot{\theta}(t)$ , the  $\mathcal{P}$  matrix would have the structure

$$\mathcal{P} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (7)$$

and the kinematic constraint equation would be

$$\begin{aligned} \mathcal{P}\dot{\omega} &= [0 \ 0 \ 0 \ \dot{\omega}_4 \ \dot{\omega}_5 \ \dot{\omega}_6]^T \\ &= [0 \ 0 \ 0 \ 0 \ 0 \ \ddot{\theta}(t)]^T \end{aligned} \quad (8)$$

where superscript  $T$  denotes the matrix transpose.

Rather than augment the  $3n$  scalar dynamic equations in Eq. (5) by the  $v$  scalar constraint equations that constitute the nontrivial part of Eq. (6), Velman (Ref. 20) uses the latter to eliminate the constraint torques  $\mathcal{L}$  from the former. To accomplish this, he notes that the vector of constraint torques (and forces)  $\mathcal{L}$  lies in the same manifold of the  $\omega$  space in which motion is prescribed. Thus if  $\mathcal{T}$  is defined as the supplementary perpendicular projection operator for  $\mathcal{P}$ , so that  $\mathcal{T} + \mathcal{P}$  is the identity matrix  $E$ ,

$$\mathcal{T} + \mathcal{P} = E \quad (9)$$

then  $\mathcal{T}$  operating on  $\mathcal{L}$  is given by

$$\mathcal{T}\mathcal{L} = 0 \quad (10)$$

For the simple example of the two hinged bodies, Eq. (10) appears in expansion as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mathcal{L}_4 \\ \mathcal{L}_5 \\ \mathcal{L}_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (11)$$

Now one can return to Eq. (5) and rewrite it as

$$\mathcal{A}(\mathcal{P} + \mathcal{T})\dot{\omega} = \mathcal{B} + \mathcal{L} \quad (12)$$

which upon premultiplication by  $\mathcal{I}$  and substitution of Eqs. (6) and (10) becomes

$$\mathcal{I}\mathcal{A}\mathcal{I}\dot{\omega} = \mathcal{I}\mathcal{B} - \mathcal{I}\mathcal{A}\mu(t) \quad (13)$$

Velman notes that the matrix  $\mathcal{I}\mathcal{A}\mathcal{I}$  is singular, so Eq. (13) is not yet ready for machine computation. Using the idempotency of the perpendicular projection operators and their supplementary character, one may write

$$\mathcal{P}\mathcal{P} = \mathcal{P}, \quad \mathcal{I}\mathcal{I} = \mathcal{I}, \quad \mathcal{P}\mathcal{I} = 0 \quad (14)$$

and then rewrite Eq. (13) in the form

$$(\mathcal{I}\mathcal{A}\mathcal{I} + \mathcal{P})\mathcal{I}\dot{\omega} = \mathcal{I}\mathcal{B} - \mathcal{I}\mathcal{A}\mu(t) \quad (15)$$

The matrix  $(\mathcal{I}\mathcal{A}\mathcal{I} + \mathcal{P})$  can be inverted to obtain

$$\mathcal{I}\dot{\omega} = (\mathcal{I}\mathcal{A}\mathcal{I} + \mathcal{P})^{-1} \mathcal{I}[\mathcal{B} - \mathcal{A}\mu(t)] \quad (16)$$

which when added to Eq. (6) yields

$$\dot{\omega} = (\mathcal{I}\mathcal{A}\mathcal{I} + \mathcal{P})^{-1} \mathcal{I}[\mathcal{B} - \mathcal{A}\mu(t)] + \mu(t) \quad (17)$$

since  $\mathcal{I} + \mathcal{P} = E$  from Eq. (9). This is the form of the equation used by Velman. The dimension of the unknown matrix  $\dot{\omega}$  is still  $3n$ , but the constraint torques have been eliminated.

As noted earlier, the fact that Velman's variables in  $\omega$  are generally relative motions simplifies the physical interpretation of the constraint equation (6), and often results in constant perpendicular projection operators  $\mathcal{P}$  and  $\mathcal{I}$ . Still, the coefficient matrix  $\mathcal{A}$  depends on the equation variables and accordingly varies with time, so the  $3n$  by  $3n$  matrix  $(\mathcal{I}\mathcal{A}\mathcal{I} + \mathcal{P})$  must be inverted at each integration step for the dynamic equation (17). Since the system has only  $3n - \nu$  degrees of freedom, one might hope to reduce the dimension of the variable  $\dot{\omega}$  in Eq. (17), and in the process reduce the size of the matrix requiring inversion at each integration step.

Fleischer (Ref. 18) has developed a procedure for accomplishing the indicated reduction of the dimension of the problem to  $3n - \nu$ . If the variable  $\omega$  is partitioned as  $\omega = [\omega_f \ \omega_c]^T$ , with subscripts  $f$  and  $c$  identifying free and constrained coordinates, then one may attempt to use the constraint equations to find  $\omega_c$  in terms of  $\omega_f$ , and then retain only that portion of Eq. (17) involving  $\dot{\omega}_f$  on the left side (i.e., truncate the matrix  $\dot{\omega}$  to  $\dot{\omega}_f$ ). Actually, Fleischer pursues a somewhat different path from the basic equations (5) and (6) to the final equations of mo-

tion, and in the process he introduces new operators to facilitate the reduction of dimension of the final equations. Because the appropriate modifications are fully treated in Ref. 18, they are not included here.

Fleischer's final equations have the form

$$\mathcal{A}_f \dot{\omega}_f = \mathcal{B}_f \quad (18)$$

in which  $\mathcal{A}_f$  and  $\mathcal{B}_f$  are time-varying and of dimension  $(3n - \nu)$  by  $(3n - \nu)$ . Computation requires the inversion of  $\mathcal{A}_f$  at each integration step (or the equivalent Gaussian elimination process), and since this inversion is the most time-consuming part of a major simulation program, this reduction of dimension would appear to be a substantial improvement. Yet there are additional computer operations involved in obtaining and solving the reduced equations, and differences in the detailed structure of the equations may impede computation in the reduced case, so perhaps it would be useful to discuss the relative computational advantages of the several forms of the discrete-parameter system equations. For example, buried within the matrices  $\mathcal{A}_f$  and  $\mathcal{B}_f$  in Eq. (18) there is the inverse of a matrix of dimension  $\nu$ . When  $\omega$  comprises relative motion variables (as in Velman's derivation), the matrix to be inverted is generally constant, requiring only one inversion operation. But when  $\omega$  comprises inertial rates of the various bodies, this matrix depends on time, and the matrix inverse must be obtained with each integration step. In Fleischer's work (Ref. 18), the basic dynamic equations of Hooker and Margulies (Ref. 14) are combined with a modified version of Velman's constraint elimination procedure (Ref. 20), so inertial rates are the variables and the indicated repeated inversion is required. At each step of the integration, Fleischer must invert a  $\nu$  by  $\nu$  matrix and then invert (or apply Gaussian elimination to) a matrix of dimension  $(3n - \nu)$  by  $(3n - \nu)$ , whereas Velman must invert (or apply Gaussian elimination to) one matrix of dimension  $3n$  by  $3n$ . As noted, Hooker and Margulies appear to favor working directly with the dynamic equations as augmented by the constraint equations, accepting system equations of dimension  $(3n + \nu)$  by  $(3n + \nu)$ . It seems probable that selection of an optimum approach will vary from one application to the next, but that for very complex systems the advantage would lie with the approach yielding equations of the lowest dimension.

In the derivations of Refs. 14-18 and 20, the approach has been first to write dynamic equations in which constraint forces and torques appear, and then (in some

cases) to use constraint equations to eliminate from the dynamic equations the constraint forces and torques and the redundant kinematic variables. This elimination is not accomplished by the analyst before computation, but in general is accomplished by the computer during numerical computation.

Other approaches have been devised that avoid constraint forces and torques from the outset, and thereby preclude the necessity of using computer time to eliminate constraint forces and torques and reduce the number of scalar equations to the number of degrees of freedom. In the context of the Newton-Euler formulation, the objective of avoiding constraint torques for a point-connected  $n$ -body system in a topological tree has been pursued successfully by Russell (Ref. 19). Kane and Wang (Refs. 26, 27) accomplish this objective even more generally, using a method described in Section II-C. Although neither Russell nor Kane and Wang support the concept of developing a single multipurpose computer program suitable for a wide range of vehicles, Russell does provide in Ref. 19 an explicit procedure for constructing equations for simulation of an  $n$ -body system restricted as above.

Russell adopts the nested-body concept advanced by Velman and described in Section II-A, where the 12-body system of Fig. 1 was discussed as an example. Recall that for this example one may first write the vector rotational equations of motion for body 12, then for subsystem 11 plus 12, etc. As pursued by Velman, this path does involve constraint torques.

Following Russell's approach, the analyst must observe that body 12 is connected to body 11 by a line hinge, and then write for body 12 only that component of the rotational equation  $\mathbf{T}^{12} = \dot{\mathbf{H}}^{12}$  paralleling the hinge axis. This avoids the constraint torques (assuming unspecified rotation about this axis), and it yields one scalar equation corresponding to the single degree of freedom of body 12 relative to body 11. Because the torque  $\mathbf{T}^{12}$  is measured relative to the mass center of body 12, the interaction force applied to body 12 by body 11 contributes to  $\mathbf{T}^{12}$ , and this unknown constraint force must be eliminated by use of the translational equation  $\mathbf{F}^{12} = m_{12} \mathbf{a}^{12}$  (Ref. 19). The method advanced by Russell retains the components of the inertial angular momenta of the individual bodies as the unknowns, obtaining angular velocity of the  $i$ th body as necessary from the matrix relationship

$$\omega^i = (I^i)^{-1} H^i \quad (19)$$

(For this reason Russell calls his method the *momentum approach*.) This distinction is central to the computational question, but not critical to the avoidance of constraint torques.

Having obtained a single scalar equation for the component of  $\mathbf{H}^{12}$  along the hinge axis, one next proceeds to consider the subsystem of bodies 11 and 12, writing translational and rotational equations as previously. Now the analyst must inspect the single joint between this subsystem and the rest of the vehicle, namely, the joint connecting bodies 11 and 10. If this is a 3-degree-of-freedom joint, the entire vector rotational equation of the subsystem 11 plus 12 is retained, but if this is a rotationally constrained joint, only those components of the vector equation that do not introduce constraint torques are retained. Whatever the nature of the joint, the number of scalar equations added corresponds to the number of degrees of freedom added in considering a new subsystem. Proceeding in this way, following the pattern of selecting subsystems previously described, one can systematically accumulate as many dynamic equations as there are independent unknowns without introducing constraint torques at all. A simple accounting procedure permits the distribution of the angular momentum of any subsystem among its constituent bodies, and from these individual inertial angular momenta the inertial angular velocities follow from Eq. (19). The dynamic equations then combine with kinematic equations just as they do for the alternative Newton-Euler formulations (Refs. 14-18, 20).

### C. Generalized-Force Methods

Among the many methods employing generalized forces, most are within the framework of Lagrange's equation

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = Q_i, \quad i = 1, \dots, n_f \quad (20)$$

where  $q_1, \dots, q_{n_f}$  are a complete and *independent* set of generalized coordinates, the Lagrangian  $L$  is the difference in kinetic and potential energy, and the generalized force  $Q_i$  is defined in terms of applied forces  $\mathbf{F}^1, \dots, \mathbf{F}^s$  and their inertial position vectors  $\mathbf{r}^1, \dots, \mathbf{r}^s$  by

$$Q_i = \sum_{j=1}^s \mathbf{F}^j \cdot \frac{\partial \mathbf{r}^j}{\partial q_i}, \quad i = 1, \dots, n_f \quad (21)$$

Lagrange's equation in the form of Eq. (20) is too familiar to warrant review and too limited to warrant adoption as the multipurpose procedure for discrete-parameter system simulation. As noted in the Introduction, restriction

to a complete and independent set of generalized coordinates  $q_1, \dots, q_{n_f}$  is inhibiting in that it precludes the use of redundant, singularity-free attitude variables and excludes nonholonomic systems. (A holonomic system can be fully described by a set of coordinates  $q_1, \dots, q_N$ , which are related by  $\nu$  equalities of the form  $f_i(q_1, \dots, q_N, t) = 0$ ,  $i = 1, \dots, \nu$ , thus permitting, at least theoretically, the determination of  $\nu$  coordinates in terms of the remaining set of  $n_f = N - \nu$  coordinates.)

The generalized forces  $Q_i$  are attractive, however, in that the definition in Eq. (21) serves to eliminate from  $Q_i$  any nonworking constraint forces, and thus to eliminate these unknown and unwanted quantities from the equations of motion.

Kane and Wang (Refs. 26, 27) have devised an approach that retains generalized forces with this desirable property, but permits application to certain nonholonomic systems and to systems with a wider range of acceptable coordinates.

Consider a dynamic system described by a complete but redundant set of coordinates  $q_1, \dots, q_N$  subject to the  $\nu$  constraint equations typified by

$$\sum_{i=1}^N A_{ji} \dot{q}_i + B_j = 0, \quad j = 1, \dots, \nu \quad (22)$$

where  $A_{ji}$  and  $B_j$  are functions of  $q_1, \dots, q_N$  and  $t$ . If Eq. (22) cannot be integrated to obtain constraint equations in the form of equalities not involving generalized velocities, this is typical of a class of nonholonomic systems.

In application to a system of  $n$  particles and rigid bodies, one may without restriction write the Newton-Euler equations for the  $i$ th body of the system in the form

$$\mathbf{F}^i + \mathbf{F}^{i*} = 0, \quad (23)$$

$$\mathbf{T}^i + \mathbf{T}^{i*} = 0, \quad (24)$$

where

$$\mathbf{F}^{i*} = -m_i \mathbf{a}^i \quad (25)$$

and

$$\mathbf{T}^{i*} = -\dot{\mathbf{H}}^i \quad (26)$$

with  $\mathbf{T}^i$ ,  $\mathbf{a}^i$ , and  $\mathbf{H}^i$  referenced to the mass center for the rigid bodies. In application to a rigid body,  $\mathbf{F}^i$  is the resultant applied force. Because the constraints imposed

in Eq. (21) imply the presence of constraint forces in  $\mathbf{F}^i$  and constraint torques in  $\mathbf{T}^i$ , these vector equations of motion when applied directly do not have the constraint elimination feature of the Lagrange Eq. (20). The objective of Kane and Wang is to modify Eqs. (23) and (24) so as to secure this advantage without accepting the noted limitations inherent in Lagrange's equation in the form of Eq. (20).

To apply the Kane-Wang method, one must first generate the necessary kinematic quantities and record the inertial forces  $\mathbf{F}^{i*}$  and inertial torques  $\mathbf{T}^{i*}$  for all bodies of the system. This process includes the derivation of the inertial linear and angular velocities  $\mathbf{v}^i$  and  $\boldsymbol{\omega}^i$  of all bodies of the system, and their expression in terms of the constrained generalized coordinates  $q_1, \dots, q_N$  and their first derivatives. It is always possible to use the  $\nu$  constraint equations given by Eq. (22) to write all velocities  $\mathbf{v}^i$  and  $\boldsymbol{\omega}^i$  in terms of  $N - \nu$  (say, the first  $N - \nu$ ) generalized velocities  $\dot{q}_1, \dots, \dot{q}_{N-\nu}$ , retaining in general all  $N$  coordinates  $q_1, \dots, q_N$  in these expressions.

The next step is the selection (by inspection of the expressions for  $\mathbf{v}^i$  and  $\boldsymbol{\omega}^i$ ) of  $N - \nu$  variables  $u_1, \dots, u_{N-\nu}$  so that (1) each  $u_i$  may be written as a linear combination of the  $N - \nu$  generalized velocities  $\dot{q}_1, \dots, \dot{q}_{N-\nu}$  plus a residual term free of generalized velocities, i.e.,

$$u_i = \sum_{j=1}^{N-\nu} U_{ij} \dot{q}_j + U_i, \quad i = 1, \dots, N - \nu \quad (27)$$

(where  $U_{ij}$  and  $U_i$  depend on  $q_1, \dots, q_N$  and  $t$ ), and (2) each velocity  $\mathbf{v}^i$  and angular velocity  $\boldsymbol{\omega}^i$  may similarly be written as a linear combination of the variables  $u_1, \dots, u_{N-\nu}$ , i.e.,

$$\mathbf{v}^i = \sum_{j=1}^{N-\nu} \mathbf{v}_j^i u_j + \mathbf{v}_i^i \quad (28)$$

and

$$\boldsymbol{\omega}^i = \sum_{j=1}^{N-\nu} \boldsymbol{\omega}_j^i u_j + \boldsymbol{\omega}_i^i \quad (29)$$

This step clearly involves some judgment on the part of the analyst, but in specific applications the choice of variables  $u_i$  is not difficult. It is always possible to choose  $u_i = \dot{q}_i$ ,  $i = 1, \dots, N - \nu$ , but this is rarely the most attractive choice. More often a judicious choice of angular velocity measure numbers for  $u_1, \dots, u_{N-\nu}$  is useful. Examples in Refs. 26 and 27 are helpful in establishing a rationale for this selection.

From this point, the application of the method is quite routine. The vectors  $\mathbf{v}_1^i, \mathbf{\omega}_1^i, \mathbf{v}_2^i, \dots, \mathbf{v}_{N-v}^i, \mathbf{\omega}_1^i, \dots, \mathbf{\omega}_{N-v}^i$  ( $i = 1, \dots, n$ ) are defined by the selection of variables  $u_1, \dots, u_{N-v}$  in Eqs. (28) and (29). These vectors are now used to define the "generalized active forces"

$$K_j = \sum_{i=1}^n (\mathbf{v}_j^i \cdot \mathbf{F}^i + \mathbf{\omega}_j^i \cdot \mathbf{T}^i), \quad j = 1, \dots, N-v \quad (30)$$

and the "generalized inertia forces"

$$K_j^* = \sum_{i=1}^n (\mathbf{v}_j^i \cdot \mathbf{F}^{i*} + \mathbf{\omega}_j^i \cdot \mathbf{T}^{i*}), \quad j = 1, \dots, N-v \quad (31)$$

Kane's theorem (proven in Ref. 26) has the form

$$K_j + K_j^* = 0, \quad j = 1, \dots, N-v \quad (32)$$

Kane shows that the basic first-order dynamic equation (32) does not involve the unknown forces and torques required to maintain the constraints of Eq. (22). Thus the combination of the  $N-v$  dynamic equations of Eq. (32), the  $v$  constraint equations of Eq. (22), and the  $N-v$  kinematic equations of Eq. (27) constitutes a complete set of  $2N-v$  first-order differential equations. This is the minimum number of equations consistent with the selection of a system of  $N$  generalized coordinates subject to  $v$  constraints. In application to the point-connected sets of  $n$  rigid bodies of Section II-B,  $N$  is  $3n$ , so Kane's equations are the same in number as those obtained by Russell's approach (Ref. 19), and the same as those actually integrated in Fleischer's program (Ref. 18). If the constraint equations are holonomic (so Eq. 22 is integrable), it is possible to replace  $v$  of the first-order differential equations by algebraic equations. When these can be solved explicitly for  $v$  coordinates in terms of the  $N-v$  remaining coordinates, the results can be substituted into the dynamic equation (32) and the kinematic equation (27) to obtain a minimum set of  $2(N-v)$  first-order differential equations.

#### D. Summary

The objective here has been to outline the several approaches to the formulation of equations of motion of discrete-parameter systems in sufficient detail to develop some appreciation of the differences in points of view, and to stimulate interest in deeper inquiry. For a comprehensive treatment of the subject, the reader is referred to the cited works.

It is impossible to make a definitive judgment of the relative merits of the many available approaches. It is difficult to weigh computer time against analyst time, particularly in the face of absolute time constraints and limitations of personnel assigned to a given task at a given time. Even with a given "cost function," the optimum approach varies from one dynamic system to the next.

Within the framework of fully programmed multipurpose digital computer programs, which require the minimum of analyst time and thought, the work of Fleischer (Ref. 18) is perhaps most useful, since it combines many of the attractive features of the earlier works of Hooker and Margulies (Ref. 14), Roberson and Wittenberg (Refs. 15, 16), and Velman (Ref. 20). It is possible that Fleischer's program would run faster if relative motion coordinates were employed, but this would introduce the judgment of the analyst into the simulation process.

There may be some question concerning the desirability in some organizations of developing a multipurpose program. Russell (Ref. 19) and Kane and Wang (Refs. 26 and 27) offer procedures that seem to lead to more efficient simulations than are available with the more highly formalized methods cited above, but they require individual programming of each problem by a capable analyst, which may be an unacceptable constraint.

### III. Hybrid-Coordinate Method

#### A. Vehicle Mathematical Model

The concept of the hybrid-coordinate method depends upon the possibility of separating a given vehicle into a number of idealized structural subsystems, each of which may be classified either as a *flexible appendage* or as a *rigid body* or *particle*. A flexible appendage is by present definition a linearly elastic structure (developing elastic stresses in proportion to strains) for which "small" deformations may be anticipated (so elastic stresses remain proportional to deformations). In most cases, internal energy dissipation may be represented by modal damping (Ref. 30), although discrete dampers can be accommodated. The usual definitions of rigid body and particle are retained.

In its present stage of development, this method is formally restricted by the requirement that each flexible appendage be attached only to a rigid body, or to more than one rigid body if those bodies are themselves interconnected in such a way that their relative motion cannot induce *deformation* of the flexible appendage. Thus the

appendage cannot properly be the only connecting structure between two rigid bodies, but it can be hinged to two bodies if their relative motion is restricted so as to permit the appendage to accommodate their relative motion by rotating without deforming. Figure 3 illustrates several examples of idealized mechanical assemblages that might present real or imagined obstacles in modeling for hybrid-coordinate analysis. In each of these sketches the structure drawn as a truss or gridwork is supposed to be substantially more flexible than the solid components.

Figure 3a bears some resemblance to a dual-spin vehicle with a flexible antenna, two flexible solar-cell arrays, and two flexible booms on the rotor. Here the de-spun platform  $B_1$ , the nutation damper mass  $B_2$ , the rotor  $B_3$ , and the contents of a spherical tank of fluid  $B_4$  are idealized as rigid. Substructures  $A^1, \dots, A^5$  are idealized as elastic appendages. An arbitrary control system determines the relative rotation of  $B_1$  and  $B_3$ , with the use of data from sensors anywhere on the vehicle. In addition, the entire antenna structure  $A^1$  may be controlled to rotate on the base  $B_1$  without exceeding the capacity of the method. But if the dish portion of  $A^1$  is controlled to rotate relative to its support tower, the model must be modified if the equations to follow are to be applied. An individual appendage is permitted only elastic deformations, so antenna dish and tower could not together be one appendage if large relative rotations are prescribed. Yet they cannot be treated as two separate flexible appendages, since each appendage must be attached to a rigid body (and not to another appendage). The only recourse is to model the tower as one or more rigid bodies, in which case the derivations of the following sections continue to apply.

Similarly, in Fig. 3b one must treat the two flexible antenna structures, together with their interconnecting rigid body, as one flexible appendage (as indicated by the dashed lines). A given flexible appendage cannot be attached to two rigid bodies for which relative motion deforms the appendage, so the massive block between the two antenna dishes cannot be isolated as a rigid body, and must instead be absorbed as part of the larger appendage A.

Again in Fig. 3c one is strictly precluded from treating each of the three blocks in the system as a rigid body; the block shown within the dashed lines must be included as a part of the appendage. Because of the large relative motions permitted between  $B_1$  and  $B_2$ , one does not have the option of including  $B_1$  plus  $B_2$  with the flexible structure instead of the block indicated. If an articulated ele-

ment such as  $B_2$  were attached as well to the block shown within the dashed lines, the hybrid-coordinate method as developed in this report would be applicable only by ignoring the masses on one end of the appendage and, instead, applying appropriate forces and torques to the end. This represents some distortion of the method, and jeopardizes the rationale to be applied to modal-coordinate truncation.

Figure 3d illustrates one of the ways in which a flexible appendage can be attached to two rigid bodies. Constraints between bodies  $B_1$  and  $B_2$  are such that their relative motion results only in the rotation (not in the deformation) of the appendage A. It would not be desirable in this case simply to include the little body  $B_2$  as part of the flexible appendage because of the discrete damper connecting  $B_2$  to  $B_1$ . The attachment between a flexible appendage and a rigid body must be consistent with the assumption of modal damping of the appendage if modal coordinates are to be obtained for second-order equations.

When two different points of a flexible structure are attached by discrete dampers (e.g., dashpots) to a rather rigid base (as in Fig. 3e), one might assume that an additional rigid body (such as  $B_2$  in Fig. 3e) connects the support points. If this assumption is unacceptable, and a discrete damping mechanism must be incorporated within the appendage, then one can find modal coordinates only after rewriting the equations of motion as first-order equations. Although modal transformations of first-order equations with arbitrary damping are treated in Section III-D, it is tentatively assumed in Section III-B that any damping in the appendage may be modeled as classic modal damping (Ref. 30).

It is evident that the equations in this section will not apply to completely general models of space vehicles, and even when applicable they may not constitute the optimum approach to simulation. For the system of Fig. 3c, for example, it may be preferable to ignore entirely the mass in the truss structure and to treat the vehicle as three discrete rigid bodies, perhaps even in a point-connected topological tree configuration if the truss is short and longitudinally stiff. A discrete-parameter simulation of the vehicle in Fig. 3a would, on the other hand, be hopelessly inefficient, and the results would be inferior to a much less expensive hybrid-coordinate simulation.

The traditional practice of normal-mode-coordinate transformation of the second-order equations of motion of the entire vehicle could not be applied directly to any of

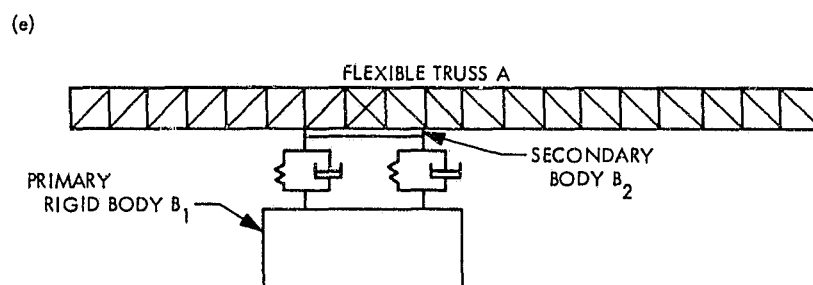
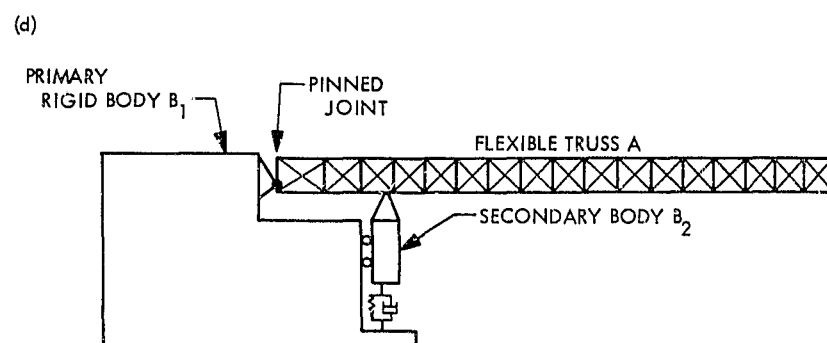
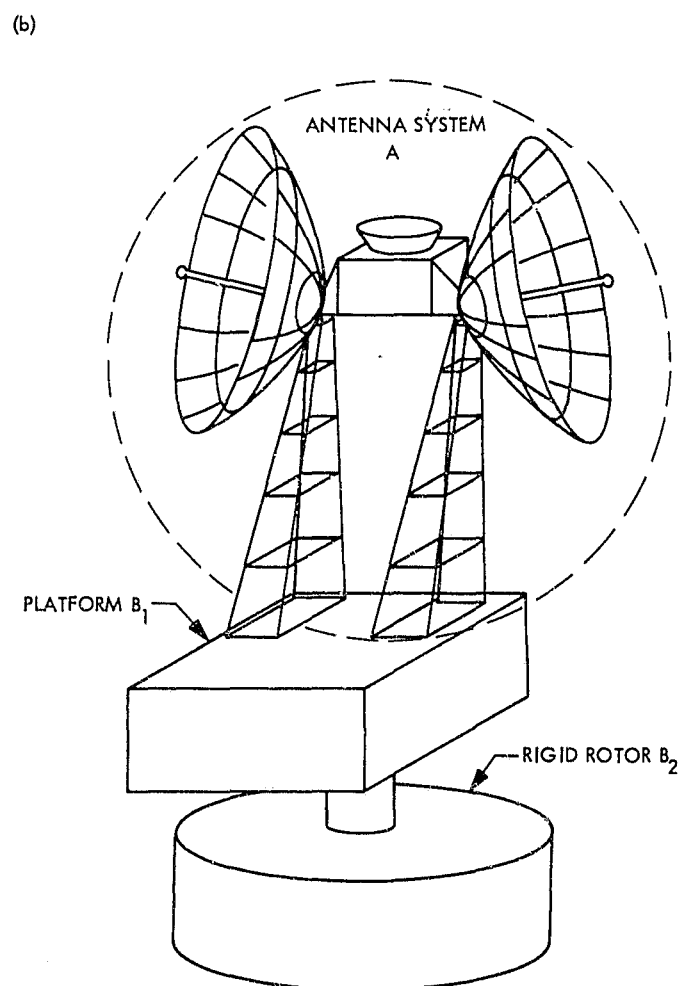
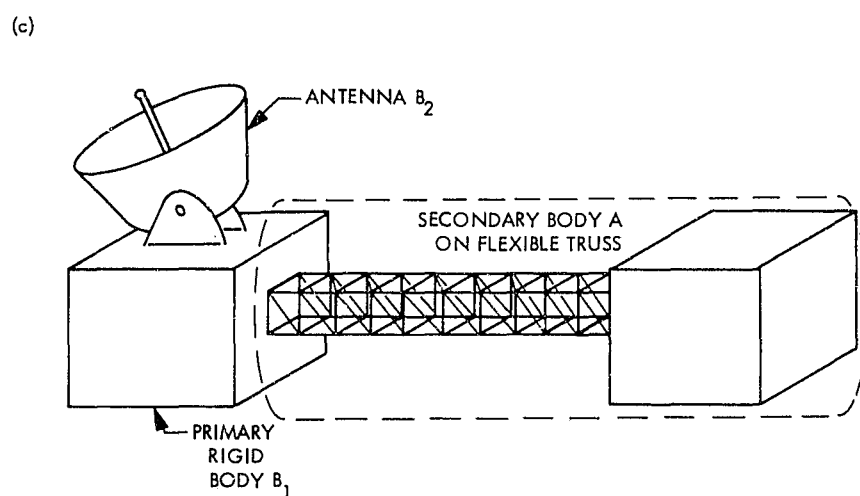
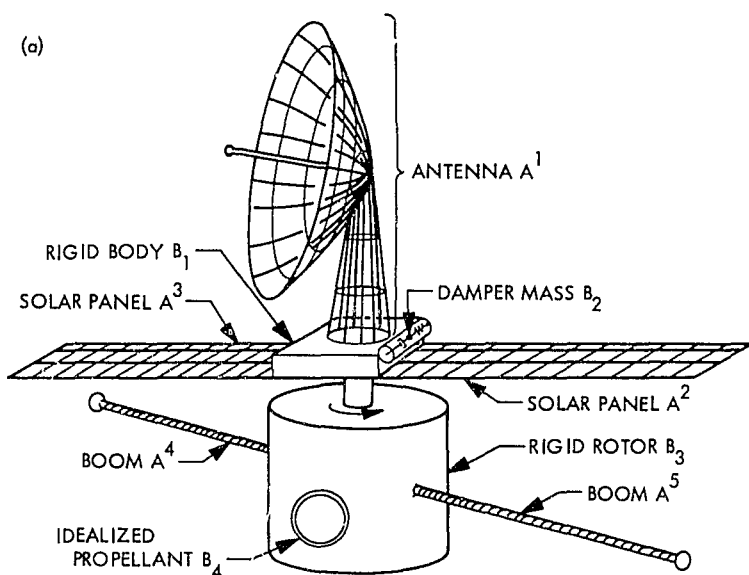


Fig. 3. Examples of models for hybrid-coordinate analysis



the vehicles of Fig. 3, although for restricted modes of motion one could obtain vehicle normal coordinates for the corresponding first-order state equations. This procedure might, for example, prove optimum for restricted motions of the mechanisms shown in Figs. 3d and 3e.

In reading the present section it may be helpful to keep Fig. 3a in mind, since this is the kind of system for which the hybrid-coordinate approach is most ideally suited.

In what follows, attention is focused first upon an individual flexible appendage  $A$  attached to a single base  $B$ . In the exceptional case (e.g., Fig. 3d), where a second rigid body is attached to the appendage, the influence of this body is felt in changing the attitude of the primary base and the appendage in its undeformed state. This attitude is reflected in a direction cosine matrix  $C$  which is considered time-dependent in the derivation.

After the individual appendage equations are examined in detail, the equations of motion of the balance of the vehicle are considered. There is no attempt to include in this report any general procedure for approaching the derivation of these equations. A class of vehicle somewhat less general than that illustrated in Fig. 3a is treated here in detail, although for more complex systems there may be some advantage in combining some of the discrete-parameter methods of Section II with the appendage equations derived explicitly here.

## B. Flexible-Appendage Equations

Consider at the outset the equations of motion of a linearly elastic structure  $A$  attached to a base  $B$ , assuming that the structure undergoes only "small" deformations while the base motion is arbitrary. ("Small" deformations means in a strict mathematical sense *arbitrarily* small or vanishing deformations, since terms above the first degree in scalar measures of deformation and deformation time derivatives are ignored completely. In engineering practice, however, this requirement is interpreted quite liberally, and a beam that suffers a tip deflection of 10% of its length is commonly described as having "small" deformations.)

For convenience in derivation and for compatibility with engineering practice, the flexible appendage  $A$  is idealized initially as a collection of elastically interconnected, discrete rigid subbodies  $A_1, \dots, A_n$ . Damping

mechanisms are excluded from this idealization,\* but modal damping is incorporated in the equations at a later stage of the derivation, with the introduction of modal coordinates for the appendage. At this point, the equations will lose their apparent restriction to discretely modeled appendages, and will be equally applicable to continuously modeled elastic appendages. In special applications to taut strings and membranes or to uniform beams, plates, or shells, the analyst may find a continuous model most convenient. In the vast majority of space applications, however, a discrete-parameter model is necessary. Frequently the model consists simply of elastically interconnected particles, but the present assumption of interconnected rigid bodies is more general and often more convenient.

Let the set of dextral orthogonal unit vectors  $b_1, b_2, b_3$  be fixed in the base  $B$ , and the similar set  $a_1, a_2, a_3$  be fixed in the reference established by  $A$  prior to deformation (i.e., fixed in that portion of  $A$  contiguous to rigid body  $B$ ). When  $A$  is elastically connected to  $B$ , these sets of vectors may be identical, and at most are related by a constant transformation matrix. But when  $A$  can rotate relative to  $B$  (as in Fig. 3a, the antenna  $A^1$  might rotate relative to the base  $B_1$ ), the transformation matrix relating these vector bases will vary with time according to an independently specified control law. If the unit vectors are written as vector arrays

$$\{a\} \equiv \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}, \quad \{b\} \equiv \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \quad (33)$$

these arrays may be treated like column matrices in recording the relationship between  $\{a\}$  and  $\{b\}$  in terms of the direction cosine matrix\*\*  $C$ , i.e.,

$$\{a\} = C \{b\} \quad (34)$$

In application it is often convenient to be able to select the basis  $\{a\}$  for each appendage individually, and to select the basis  $\{b\}$  independently (perhaps guided by the principal axes of inertia of the total vehicle, or by

\*Terms to be added to the equations of this subsection to accommodate idealized discrete dampers are easily generated, and are accommodated in the coordinate transformations of subsequent subsections.

\*\*In this report matrices are not identified by brackets [ ] unless the elements of the matrix are recorded in detail. Vector arrays are always enclosed in braces { }.



design coordinates of a drawing, or by attitude-control axes). Thus even when  $A$  does not rotate relative to  $B$ , the matrix  $C$  may not be the identity matrix  $E$ . When  $A$  does rotate relative to  $B$ , so that  $C$  varies with time, it will become necessary either to augment the equations in this section by added dynamic equations or to assume that perfect control results in known time behavior of  $C$ .

In devising the discrete-parameter model of  $A$ , it is convenient to select basis  $\{a\}$  and the individual mass-center principal axis bases of  $A_1, \dots, A_n$  to be identical.

The first step is the formulation of the equations of motion of a typical body  $A_s$  in form convenient for combination with corresponding equations for all  $n$  bodies. Consequently the vector-dyadic equations of translation and rotation will be written as matrix equations in basis  $\{a\}$ .

Let  $P_s$  be the mass center of body  $A_s$ , and let  $O'$  be a point fixed in inertial space (see Fig. 4). Then Newton's second law provides

$$\mathbf{F}^s = m_s (\ddot{\mathbf{P}}_s) \quad (35)$$

where  $\mathbf{F}^s$  is the resultant force applied to  $A_s$ ,  $m_s$  is the mass of  $A_s$ ,  $\mathbf{P}_s$  is the vector from  $O'$  to  $P_s$ , and (as noted previously) each dot over a vector denotes time differentiation in an inertial frame of reference.

As shown in Fig. 4, the inertial position vector  $\mathbf{P}_s$  may be written as

$$\mathbf{P}_s = \mathbf{X} + \mathbf{c} + \mathbf{R} + \mathbf{r}^s + \mathbf{u}^s \quad (36)$$

where  $\mathbf{X}$  is the inertial position vector of the vehicle mass center  $CM$ ;  $\mathbf{c}$  is the vector from  $CM$  to the point  $O$  fixed relative to body  $B$  and coincident with  $CM$  when the vehicle is in some nominal undeformed configuration;  $\mathbf{R}$  is the vector from  $O$  to an arbitrary point  $Q$  fixed in  $B$  on the interface between  $A$  and  $B$ ;  $\mathbf{r}^s$  is the vector locating from  $Q$  the point  $Q_s$  occupied by  $P_s$  when  $A$  is undeformed; and  $\mathbf{u}^s$  defines the translational deformation of the appendage at point  $Q_s$ . The vectors in Eq. (36) are not all expressed conveniently in any one vector basis. Vector  $\mathbf{X}$  defines the vehicle trajectory, which may be known in terms of an inertial reference. It thus becomes desirable to express  $\mathbf{X}$  in terms of an inertially fixed vector basis  $i_1, i_2, i_3$ . Vectors  $\mathbf{R}$  and  $\mathbf{r}^s$ , on the other hand, are fixed in the reference frames in which  $b_1, b_2, b_3$  and  $a_1, a_2, a_3$  are embedded, respectively, so they are most efficiently expressed in their own natural vector bases.

Since a matrix formulation is ultimately required, 3 by 1 matrices are defined for each of the vectors in Eq. (36) in terms of the appropriate vector basis. These matrices are introduced with the use of vector arrays (as in Eq. 33), by the definitions

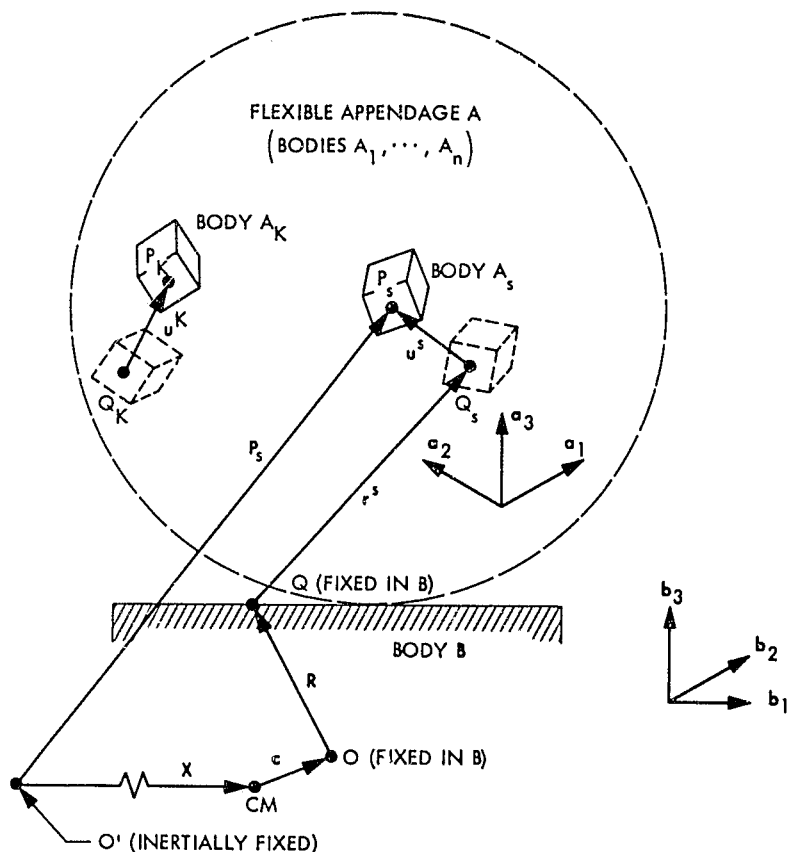
$$\begin{aligned} \mathbf{X} &= \{i_1 i_2 i_3\} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \equiv \{i\}^T \mathbf{X} \\ \mathbf{c} &= \{b_1 b_2 b_3\} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \equiv \{b\}^T \mathbf{c} \\ \mathbf{R} &= \{b_1 b_2 b_3\} \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \equiv \{b\}^T \mathbf{R} \\ \mathbf{r}^s &= \{a_1 a_2 a_3\} \begin{bmatrix} r_1^s \\ r_2^s \\ r_3^s \end{bmatrix} \equiv \{a\}^T \mathbf{r}^s \\ \mathbf{u}^s &= \{a_1 a_2 a_3\} \begin{bmatrix} u_1^s \\ u_2^s \\ u_3^s \end{bmatrix} \equiv \{a\}^T \mathbf{u}^s \end{aligned} \quad (37)$$

where the superscript  $T$  denotes the transpose of a matrix or vector array.

Upon substituting Eq. (36) into Eq. (35), one confronts the necessity of differentiating in an inertial reference frame a number of vectors that are much more easily expressed (and sometimes constant) in some other reference frame. This is most easily accomplished by liberal use of the identity from vector differential calculus

$$\frac{{}^{f_1}d}{{}^{f_2}dt} \mathbf{V} = \frac{{}^{f_2}d}{{}^{f_2}dt} \mathbf{V} + {}^{f_1}\omega^{f_2} \times \mathbf{V} \quad (38)$$

where  $\mathbf{V}$  is any vector,  ${}^{f_1}\omega^{f_2}$  is the angular velocity of any reference frame  $f_2$  relative to any other reference frame  $f_1$ , and the superscript preceding the derivative operator denotes the reference frame of differentiation. Although  $t$  may be any independent variable, in the present context it is always time.



**Fig. 4. Discrete-parameter appendage sub-body coordinates**

If the superscript  $i$  denotes the inertial reference frame, Eqs. (35) and (36) may be combined as

$$\mathbf{F}^s = m_s \left[ \frac{d^2}{dt^2} \mathbf{X} + \frac{d^2}{dt^2} (\mathbf{c} + \mathbf{R}) + \frac{d^2}{dt^2} (\mathbf{r}^s + \mathbf{u}^s) \right] \quad (39)$$

The first term in brackets in Eq. (39) is (with the use of Eq. 37) simply

$$\frac{d^2}{dt^2} (\{\mathbf{i}\}^T \mathbf{X}) = \{\mathbf{i}\}^T \ddot{\mathbf{X}} \quad (40)$$

since the inertial time derivative of  $\{\mathbf{i}\}$  is zero. (Recall that a matrix derivative such as  $\ddot{\mathbf{X}}$  is just the matrix of differentiated elements.)

The second term in brackets in Eq. (39) may be obtained (with the use of Eq. 38) from

$$\frac{d}{dt} (\mathbf{c} + \mathbf{R}) = \frac{d}{dt} (\mathbf{c} + \mathbf{R}) + \boldsymbol{\omega} \times (\mathbf{c} + \mathbf{R}) \quad (41)$$

where  $b$  is the reference frame of body  $B$  and basis  $\{\mathbf{b}\}$ , and  $\boldsymbol{\omega}$  is the inertial angular velocity of  $b$ . Differentiating Eq. (41) provides (with the use of Eq. 38 again)

$$\begin{aligned} \frac{d^2}{dt^2} (\mathbf{c} + \mathbf{R}) &= \frac{d^2}{dt^2} (\mathbf{c} + \mathbf{R}) \\ &+ 2\boldsymbol{\omega} \times \frac{d}{dt} (\mathbf{c} + \mathbf{R}) + \boldsymbol{\omega} \times [\boldsymbol{\omega} \times (\mathbf{c} + \mathbf{R})] \\ &+ \dot{\boldsymbol{\omega}} \times (\mathbf{c} + \mathbf{R}) \end{aligned} \quad (42)$$

Now Eq. (37) may be utilized, noting  $R$  to be constant, to obtain

$$\begin{aligned} \frac{d^2}{dt^2} (\mathbf{c} + \mathbf{R}) &= \{\mathbf{b}\}^T \ddot{\mathbf{c}} + 2\boldsymbol{\omega} \times \{\mathbf{b}\}^T \dot{\mathbf{c}} \\ &+ \boldsymbol{\omega} \times [\boldsymbol{\omega} \times \{\mathbf{b}\}^T (\mathbf{c} + \mathbf{R})] \\ &+ \dot{\boldsymbol{\omega}} \times \{\mathbf{b}\}^T (\mathbf{c} + \mathbf{R}) \end{aligned} \quad (43)$$

To express this result in matrix terms, one must obtain the matrix equivalent of a vector cross product.

For two arbitrary vectors  $\mathbf{V}$  and  $\mathbf{W}$ , expressed in terms of an arbitrary vector basis  $\{\mathbf{e}\}$ , the vector cross product may be written as

$$\mathbf{V} \times \mathbf{W} = \{\mathbf{e}\}^T \mathbf{V} \times \{\mathbf{e}\}^T \mathbf{W} = \{\mathbf{e}\}^T \tilde{\mathbf{V}} \mathbf{W}$$

where

$$\begin{aligned} \mathbf{V} &\equiv \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}, & \mathbf{W} &\equiv \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix}, \\ \tilde{\mathbf{V}} &\equiv \begin{bmatrix} 0 & -V_3 & V_2 \\ V_3 & 0 & -V_1 \\ -V_2 & V_1 & 0 \end{bmatrix} \end{aligned} \quad (44)$$

as may be confirmed by expansion. In equivalent terms, the matrix product  $\tilde{\mathbf{V}} \mathbf{W}$  is isomorphic to the vector product  $\mathbf{V} \times \mathbf{W}$ . Thus the matrix identity

$$\tilde{\mathbf{V}} \mathbf{W} = -\tilde{\mathbf{W}} \mathbf{V} \quad (45)$$

follows from the vector identity  $\mathbf{V} \times \mathbf{W} = -\mathbf{W} \times \mathbf{V}$ . The tilde operator ( $\sim$ ) over a 3 by 1 matrix represents the

corresponding skew-symmetric 3 by 3 matrix, as in Eq. (44), throughout this report. Note the useful identity

$$\tilde{V}^T = -\tilde{V} \quad (46)$$

for any tilded matrix, because of skew symmetry.

Equation (43) can now be written (with the definition  $\omega \equiv \{\mathbf{b}\}^T \omega$ ) as

$$\frac{{}^i d^2}{dt^2} (\mathbf{c} + \mathbf{R}) = \{\mathbf{b}\}^T [\ddot{\mathbf{c}} + 2\tilde{\omega}\dot{\mathbf{c}} + \tilde{\omega}\tilde{\omega}(\mathbf{c} + \mathbf{R}) + \tilde{\omega}(\mathbf{c} + \mathbf{R})] \quad (47)$$

which provides the second term in brackets in Eq. (39).

The final term in Eq. (39) can be obtained in similar form, although the vectors  $\mathbf{r}^s$  and  $\mathbf{u}^s$  are written in basis  $\{\mathbf{a}\}$  in Eq. (37). The differentiation then is performed in two stages. First,

$$\begin{aligned} \frac{{}^i d^2}{dt^2} (\mathbf{r}^s + \mathbf{u}^s) &= \frac{{}^b d^2}{dt^2} (\mathbf{r}^s + \mathbf{u}^s) + 2\omega \times \frac{{}^b d}{dt} (\mathbf{r}^s + \mathbf{u}^s) \\ &+ \omega \times [\omega \times (\mathbf{r}^s + \mathbf{u}^s)] + \dot{\omega} \times (\mathbf{r}^s + \mathbf{u}^s) \end{aligned}$$

is obtained from Eq. (42), and then the derivatives in  $\{\mathbf{b}\}$  are replaced by derivatives in  $\{\mathbf{a}\}$  by using Eqs. (41) and (42), and replacing  $i$  by  $b$  and  $b$  by  $a$ . The result is

$$\begin{aligned} \frac{{}^i d^2}{dt^2} (\mathbf{r}^s + \mathbf{u}^s) &= \frac{{}^a d^2}{dt^2} \mathbf{u}^s + 2\Omega^a \times \frac{{}^a d}{dt} \mathbf{u}^s \\ &+ \Omega^a \times [\Omega^a \times (\mathbf{r}^s + \mathbf{u}^s)] \\ &+ \frac{{}^b d}{dt} \Omega^a \times (\mathbf{r}^s + \mathbf{u}^s) \\ &+ 2\omega \times \left[ \frac{{}^a d}{dt} \mathbf{u}^s + \Omega^a \times (\mathbf{r}^s + \mathbf{u}^s) \right] \\ &+ \omega \times [\omega \times (\mathbf{r}^s + \mathbf{u}^s)] + \dot{\omega} \times (\mathbf{r}^s + \mathbf{u}^s) \end{aligned} \quad (48)$$

where  $\Omega^a$  is the angular velocity of the  $\{\mathbf{a}\}$  frame relative to the  $\{\mathbf{b}\}$  frame, and it has been recognized that the time derivative of  $\mathbf{r}^s$  in reference  $a$  is zero. This result is now to be written in terms of the vector array  $\{\mathbf{a}\}$  and the matrices defined in Eq. (37). If  $\Omega^a$  is written in the  $\{\mathbf{a}\}$  basis as

$$\Omega^a = \{\mathbf{a}\}^T \Omega^a \quad (49)$$

and Eq. (34) is used to write

$$\{\mathbf{b}\}^T = \{\mathbf{a}\}^T C \quad (50)$$

so that  $\omega$  is written in the  $\{\mathbf{a}\}$  basis as

$$\omega = \{\mathbf{b}\}^T \omega = \{\mathbf{a}\}^T C \omega$$

then Eq. (48) becomes

$$\begin{aligned} \frac{{}^i d^2}{dt^2} (\mathbf{r}^s + \mathbf{u}^s) &= \{\mathbf{a}\}^T [\ddot{\mathbf{u}}^s + 2\tilde{\Omega}^a \dot{\mathbf{u}}^s + \tilde{\Omega}^a \tilde{\Omega}^a (\mathbf{r}^s + \mathbf{u}^s) + \tilde{\Omega}^a (\mathbf{r}^s + \mathbf{u}^s) \\ &+ 2(\tilde{C}\omega) \dot{\mathbf{u}}^s + 2(\tilde{C}\omega) \tilde{\Omega}^a (\mathbf{r}^s + \mathbf{u}^s) + (\tilde{C}\omega)(\tilde{C}\omega) (\mathbf{r}^s + \mathbf{u}^s) + (\tilde{C}\dot{\omega}) (\mathbf{r}^s + \mathbf{u}^s)] \end{aligned} \quad (51)$$

When Eqs. (51), (47), and (40) are combined in Eq. (39), the three different vector bases of these equations are abandoned in favor of basis  $\{\mathbf{a}\}$ , noting that this is also the natural basis for  $\mathbf{F}^s$ . This requires the use of Eq. (50) in Eq. (47), and the parallel use in Eq. (40) of the relationship

$$\{\mathbf{i}\}^T = \{\mathbf{b}\}^T \Theta = \{\mathbf{a}\}^T C \Theta \quad (52)$$

where  $\Theta$  is the direction cosine matrix relating the body  $B$  to an inertially fixed vector basis, i.e.,

$$\{\mathbf{b}\} = \Theta \{\mathbf{i}\} \quad (53)$$

The result of these substitutions into Eq. (39) is the vector equation

$$\begin{aligned} \{\mathbf{a}\}^T \mathbf{F}^s &= \{\mathbf{a}\}^T m_s \{C\Theta\ddot{\mathbf{X}} + C[\ddot{\mathbf{c}} + 2\tilde{\omega}\dot{\mathbf{c}} + \tilde{\omega}\tilde{\omega}(\mathbf{c} + \mathbf{R}) + \tilde{\omega}(\mathbf{c} + \mathbf{R})] + \ddot{\mathbf{u}}^s + 2\tilde{\Omega}^a \dot{\mathbf{u}}^s + \tilde{\Omega}^a \tilde{\Omega}^a (\mathbf{r}^s + \mathbf{u}^s) \\ &+ \tilde{\Omega}^a (\mathbf{r}^s + \mathbf{u}^s) + 2(\tilde{C}\omega) \dot{\mathbf{u}}^s + 2(\tilde{C}\omega) \tilde{\Omega}^a (\mathbf{r}^s + \mathbf{u}^s) + (\tilde{C}\omega)(\tilde{C}\omega) (\mathbf{r}^s + \mathbf{u}^s) + (\tilde{C}\dot{\omega}) (\mathbf{r}^s + \mathbf{u}^s)\} \end{aligned} \quad (54)$$

which employs the new matrix symbol  $F^s$  as defined by

$$\mathbf{F}^s \equiv \{\mathbf{a}\}^T F^s \quad (55)$$

One may now record the corresponding matrix equation (in vector basis  $\{\mathbf{a}\}$ ), either by inspection of Eq. (54), or by formal dot-multiplication by the vector array  $\{\mathbf{a}\}$ , noting that the relationship

$$\{\mathbf{a}\} \cdot \{\mathbf{a}\}^T \equiv \begin{Bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{Bmatrix} \cdot \{\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3\} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \equiv E \quad (56)$$

follows from the introduction of the dot between vector arrays to imply the multiplication of vector arrays following the rules of matrix multiplication, but with dot-multiplication of the vector elements inside the arrays. This notation is sometimes convenient for notational bookkeeping, but even without this artifice one can simply record the matrix equation isomorphic to the vector equation (54). In the process, the future direction of the derivation may be anticipated by the replacement of the scalar  $m_s$  with the matrix  $m^s = m_s E$ , with  $E$  the 3 by 3 identity matrix. The desired matrix equation is then obtained (after some factoring and rearrangement) in the form

$$F^s = m^s \{ C [\ddot{\theta} \dot{X} + \ddot{c} + 2\tilde{\omega}\dot{c} + \tilde{\omega}\tilde{\omega}(c + R) - (\tilde{c} + \tilde{R})\dot{\omega}] - (\tilde{r}^s + \tilde{u}^s)(\dot{\Omega}^a + C\dot{\omega}) + [(\tilde{C}_\omega)(\tilde{C}_\omega) + 2(\tilde{C}_\omega)(\tilde{\Omega}^a) + \tilde{\Omega}^a \tilde{\Omega}^a](r^s + u^s) + 2[\tilde{\Omega}^a + (\tilde{C}_\omega)]\dot{u}^s + \ddot{u}^s \} \quad (57)$$

Of course the matrix  $c$ , which denotes the translation in body  $B$  of the vehicle mass center, depends in part on the translations  $u^s$  ( $s = 1, \dots, n$ ) of the sub-bodies of the appendage  $A$ . In general, it may also depend on the deformations of other elastic appendages of the system, and on the (possibly large) relative motions of other articulated rigid bodies of the vehicle. To emphasize the influence on  $c$  of the deformation of the appendage for which equations of motion are being written, the mass-center definition is employed to write for  $c$

$$c = -\frac{1}{\mathcal{M}} \sum_{s=1}^n m_s u^s + e \quad (58)$$

where  $e$  accommodates the mass-center motion due to other appendages and moving parts, if any exist, and  $\mathcal{M}$  is the total vehicle mass. With the definition

$$e \equiv \{\mathbf{b}\}^T e \quad (59)$$

Eq. (58) may be written in the form

$$\begin{aligned} \{\mathbf{b}\}^T c &= -\frac{1}{\mathcal{M}} \sum_{s=1}^n \{\mathbf{a}\}^T m^s u^s + \{\mathbf{b}\}^T e \\ &= \{\mathbf{b}\}^T [-\sum_{s=1}^n C^T \mu^s u^s + e] \end{aligned}$$

where  $\mu^s \equiv m^s / \mathcal{M}$  and Eq. (34) has been employed. This vector equation has the matrix counterpart

$$c = -\sum_{s=1}^n C^T \mu^s u^s + e \quad (60)$$

Thus the role of the appendage deformation in Eq. (57) is made more explicit by the substitution of Eq. (60) and its derivatives (permitting  $C$  to vary with time), which results in

$$\dot{c} = -\sum_{s=1}^n (\dot{C}^T \mu^s u^s + C^T \mu^s \dot{u}^s) + \dot{e} \quad (61)$$

and

$$\ddot{c} = -\sum_{s=1}^n (\ddot{C}^T \mu^s u^s + 2\dot{C}^T \mu^s \dot{u}^s + C^T \mu^s \ddot{u}^s) + \ddot{e} \quad (62)$$

If it seems preferable to avoid the time derivatives of the direction cosine matrix  $C$ , this can be accomplished by using the identity (38) once more to obtain

$$\begin{aligned} \{\mathbf{b}\}^T \dot{c} &= \frac{b d}{dt} [\{\mathbf{b}\}^T c] \\ &= \frac{b d}{dt} [-\{\mathbf{a}\}^T \sum_{s=1}^n \mu^s u^s + \{\mathbf{b}\}^T e] \\ &= \frac{a d}{dt} [-\{\mathbf{a}\}^T \sum_{s=1}^n \mu^s u^s] \\ &\quad + \Omega^a \times [-\{\mathbf{a}\}^T \sum_{s=1}^n \mu^s u^s] + \{\mathbf{b}\}^T \dot{e} \\ &= -\{\mathbf{a}\}^T \sum_{s=1}^n \mu^s \dot{u}^s - \{\mathbf{a}\}^T \tilde{\Omega}^a \sum_{s=1}^n \mu^s u^s + \{\mathbf{b}\}^T \dot{e} \end{aligned} \quad (63)$$

With Eq. (34), this yields the matrix equation

$$\dot{C} = -\sum_{s=1}^n C^T \mu^s \dot{u}^s - C^T \tilde{\Omega}^a \sum_{s=1}^n \mu^s u^s + \dot{e} \quad (64)$$

Differentiation of Eq. (63) in reference frame  $b$  similarly provides the vector equation

$$\{b\}^T \ddot{e} = -\{a\}^T \left[ \sum_{s=1}^n \mu^s \ddot{u}^s + \tilde{\Omega}^a \sum_{s=1}^n \mu^s u^s + \tilde{\Omega}^a \sum_{s=1}^n \mu^s \dot{u}^s \right] - \{a\}^T \tilde{\Omega}^a \left[ \sum_{s=1}^n \mu^s \dot{u}^s + \tilde{\Omega}^a \sum_{s=1}^n \mu^s u^s \right] + \{b\}^T \ddot{e}$$

The matrix counterpart in basis  $\{b\}$  for this vector equation is

$$\ddot{e} = -C^T \left[ \sum_{s=1}^n \mu^s \ddot{u}^s + \tilde{\Omega}^a \sum_{s=1}^n \mu^s u^s + 2\tilde{\Omega}^a \sum_{s=1}^n \mu^s \dot{u}^s + \tilde{\Omega}^a \tilde{\Omega}^a \sum_{s=1}^n \mu^s u^s \right] + \ddot{e} \quad (65)$$

Now Eqs. (64) and (65) may be used in Eq. (57), instead of substituting Eqs. (61) and (62) in this equation, thereby avoiding derivatives of  $C$ . Alternatively, one can avoid the relative angular rate  $\Omega^a$  and its time derivatives entirely by the substitution of the kinematic identity (Ref. 16)

$$\tilde{\Omega}^a = C \dot{C}^T \quad (66)$$

and its consequence

$$\ddot{\Omega}^a = \dot{C} \dot{C}^T + C \ddot{C}^T \quad (67)$$

With these substitutions in Eq. (57), and with Eqs. (61) and (62), all relative motion between bases  $\{a\}$  and  $\{b\}$  is expressed in terms of the direction cosine matrix  $C$  and its derivatives. Although this option may be optimum in some applications, it would seem preferable in most practical cases to avoid  $\dot{C}$  and  $\ddot{C}$ , and therefore to employ Eqs. (57), (64), and (65).

Since the sub-bodies  $A_1, \dots, A_n$  are considered to be rigid bodies rather than particles, the translational equation (57) must be augmented by rotational equations such as

$$\mathbf{T}^s = \dot{\mathbf{H}}^s \quad (68)$$

for typical rigid sub-body  $A_s$ . Here  $\mathbf{H}^s$  is the inertial angular momentum of  $A_s$  referred to the mass center  $P_s$ , and  $\mathbf{T}^s$  is the corresponding resultant torque. The rules of vector differentiation (see Eq. 38) may now be applied to  $\mathbf{H}^s$ , or to its equivalent  $\mathbf{I}^s \cdot \boldsymbol{\omega}^s$ , where  $\mathbf{I}^s$  is the mass-center inertia dyadic of  $A_s$  and  $\boldsymbol{\omega}^s$  the inertial angular velocity of  $A_s$ . The result is

$$\begin{aligned} \mathbf{T}^s &= \frac{d}{dt} (\mathbf{H}^s) \\ &= \frac{d}{dt} (\mathbf{I}^s \cdot \boldsymbol{\omega}^s) = \mathbf{I}^s \cdot \frac{d}{dt} \boldsymbol{\omega}^s + \frac{d}{dt} \mathbf{I}^s \cdot \boldsymbol{\omega}^s \\ &= \mathbf{I}^s \cdot \frac{d}{dt} \boldsymbol{\omega}^s + \left( \frac{d}{dt} \mathbf{I}^s + \boldsymbol{\omega}^s \times \mathbf{I}^s - \mathbf{I}^s \times \boldsymbol{\omega}^s \right) \cdot \boldsymbol{\omega}^s \end{aligned} \quad (69)$$

where the symbol  $^s d/dt$  denotes differentiation in the reference frame of  $A_s$ . Here use has been made of the counterpart for dyadic differentiation of the vector differentiation relationship of Eq. (38), namely

$$\frac{f_1 d}{dt} \mathbf{D} = \frac{f_2 d}{dt} \mathbf{D} + f_1 \boldsymbol{\omega}^{f_2} \times \mathbf{D} - \mathbf{D} \times f_1 \boldsymbol{\omega}^{f_2} \quad (70)$$

with  $\mathbf{D}$  any dyadic and  $f_1$  and  $f_2$  any two frames of reference. (This relationship can be confirmed by writing the dyadic in the form\*  $\mathbf{D} = D_{\alpha\beta} \mathbf{e}_\alpha \mathbf{e}_\beta$ , and then applying Eq. (38) to the unit vectors  $\mathbf{e}_\alpha$  and  $\mathbf{e}_\beta$ .)

The term  $\mathbf{I}^s \times \boldsymbol{\omega}^s \cdot \boldsymbol{\omega}^s$  in Eq. (69) is zero, since the implied operations include dot-multiplying by  $\boldsymbol{\omega}^s$  a vector orthogonal to  $\boldsymbol{\omega}^s$ . Equation (69) is further simplified by the constancy of the inertia dyadic of the rigid body  $A_s$  in the reference frame of that body, i.e.,  $(^s d/dt) \mathbf{I}^s$  is zero. In rewriting the correspondingly simplified rotational equation, one may expand the inertial angular velocity  $\boldsymbol{\omega}^s$  of  $A_s$  by use of the "chain rule"

$$f_1 \boldsymbol{\omega}^{f_1} = f_1 \boldsymbol{\omega}^{f_2} + f_2 \boldsymbol{\omega}^{f_3} + \dots + f_{n-1} \boldsymbol{\omega}^{f_n} \quad (71)$$

\*Throughout this report, lower-case Greek indices range in value from 1 to 3, and when these indices are repeated in a given term, summation over these values is implied.

where  $f_1, \dots, f_n$  are  $n$  arbitrary frames of reference and  ${}^{f_s}\omega^{f_r}$  is the angular velocity of frame  $f_r$  relative to frame  $f_s$ . In this case the chain is from the frame of body  $A_s$  to the frame of  $A$ , to the frame of  $B$ , to inertial space. Since the rotation of  $A_s$  relative to  $A$  is due only to "small" structural deformations, this rotation can be represented by the vector  $\beta^s = \beta_1^s \mathbf{a}_1 + \beta_2^s \mathbf{a}_2 + \beta_3^s \mathbf{a}_3$ , where  $\beta_1^s, \beta_2^s, \beta_3^s$  are three angles of rotation about orthogonal axes  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ . This representation is strictly applicable only if these rotations are infinitesimal, but this is nonetheless the commonplace assumption of structural dynamicists. Consequently  $\omega^s$  in Eq. (69) can be replaced by the use of Eq. (71) to write

$$\omega^s = \omega + \Omega^a + \{\mathbf{a}\}^T \dot{\beta}^s \quad (72)$$

where  $\beta^s$  is the 3 by 1 matrix defined by  $\beta^s = \{\mathbf{a}\}^T \beta^s$ . This substitution puts Eq. (69) in the form

$$\begin{aligned} \mathbf{T}^s = \mathbf{I}^s \cdot \frac{d}{dt} (\omega + \Omega^a + \{\mathbf{a}\}^T \dot{\beta}^s) \\ + (\omega + \Omega^a + \{\mathbf{a}\}^T \dot{\beta}^s) \times \mathbf{I}^s \cdot (\omega + \Omega^a + \{\mathbf{a}\}^T \dot{\beta}^s) \end{aligned} \quad (73)$$

Again the differentiation is accomplished with the help of Eq. (38), and simplification is afforded by the assumed smallness of  $\dot{\beta}^s$  (since only first-degree terms in  $\dot{\beta}^s$  and  $\ddot{\beta}^s$  are retained). The result may be written as

$$\begin{aligned} \mathbf{T}^s = \mathbf{I}^s \cdot \left[ \frac{d}{dt} \omega + \frac{d}{dt} \Omega^a + (\omega + \Omega^a) \times \Omega^a + \{\mathbf{a}\}^T \ddot{\beta}^s + (\omega + \Omega^a) \times \{\mathbf{a}\}^T \dot{\beta}^s \right] \\ + (\omega + \Omega^a) \times \mathbf{I}^s \cdot (\omega + \Omega^a) + (\omega + \Omega^a) \times \mathbf{I}^s \cdot \{\mathbf{a}\}^T \dot{\beta}^s + \{\mathbf{a}\}^T \dot{\beta}^s \times \mathbf{I}^s \cdot (\omega + \Omega^a) \end{aligned} \quad (74)$$

If all vectors are now written in terms of the vector arrays of the most convenient basis, with the use of the definitions following Eq. (49) together with

$$\mathbf{T}^s \equiv \{\mathbf{a}\}^T T^s$$

and, with the summation convention,

$$\mathbf{I}^s \equiv I_{\alpha\beta}^s \mathbf{a}_\alpha \mathbf{a}_\beta \equiv \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} \begin{bmatrix} I_{11}^s & I_{12}^s & I_{13}^s \\ I_{21}^s & I_{22}^s & I_{23}^s \\ I_{31}^s & I_{32}^s & I_{33}^s \end{bmatrix} \begin{Bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{Bmatrix} \equiv \{\mathbf{a}\}^T I^s \{\mathbf{a}\} \quad (75)$$

the vector rotational equation becomes

$$\begin{aligned} \{\mathbf{a}\}^T T^s = \{\mathbf{a}\}^T I^s \{\mathbf{a}\} \cdot [\{\mathbf{b}\}^T \dot{\omega} + \{\mathbf{a}\}^T \dot{\Omega}^a + \{\mathbf{b}\}^T \omega \times \{\mathbf{a}\}^T \Omega^a + \{\mathbf{a}\}^T \ddot{\beta}^s + (\{\mathbf{b}\}^T \omega + \{\mathbf{a}\}^T \Omega^a) \times \{\mathbf{a}\}^T \dot{\beta}^s] \\ + (\{\mathbf{b}\}^T \omega + \{\mathbf{a}\}^T \Omega^a) \times \{\mathbf{a}\}^T I^s \{\mathbf{a}\} \cdot (\{\mathbf{b}\}^T \omega + \{\mathbf{a}\}^T \Omega^a) \\ + (\{\mathbf{b}\}^T \omega + \{\mathbf{a}\}^T \Omega^a) \times \{\mathbf{a}\}^T I^s \{\mathbf{a}\} \cdot \{\mathbf{a}\}^T \dot{\beta}^s + \{\mathbf{a}\}^T \dot{\beta}^s \times \{\mathbf{a}\}^T I^s \{\mathbf{a}\} \cdot (\{\mathbf{b}\}^T \omega + \{\mathbf{a}\}^T \Omega^a) \end{aligned} \quad (76)$$

By using Eq. (50) to obtain every vector in basis  $\{\mathbf{a}\}$ , and applying the matrix representations of vector cross-multiplication (see Eq. 44) and vector array dot-multiplication (Eq. 56), one can obtain this vector equation in the useful form

$$\begin{aligned} \{\mathbf{a}\}^T T^s = \{\mathbf{a}\}^T \{I^s [C\dot{\omega} + \dot{\Omega}^a + (\tilde{C}\omega) \Omega^a + \ddot{\beta}^s + (\tilde{C}\omega) \dot{\beta}^s + \tilde{\Omega}^a \dot{\beta}^s] \\ + [(\tilde{C}\omega) + \tilde{\Omega}^a] I^s [C\omega + \Omega^a] + [(\tilde{C}\omega) + \tilde{\Omega}^a] I^s \dot{\beta}^s + \tilde{\beta}^s I^s [C\omega + \Omega^a]\} \end{aligned} \quad (77)$$

Now the isomorphic matrix equation can be written by inspection (or obtained formally by dot-multiplying by  $\{\mathbf{a}\}$ ). In the process, the identity of Eq. (45) is applied whenever it seems convenient to have the most obviously unknown variable available for factoring on the extreme right. For example, in the last term the unknown deformation  $\dot{\beta}^s$  must appear on the right if it is to be factored out with the term preceding it, so the identity

$$\tilde{\beta}^s I^s (C\omega + \Omega^a) = -[I^s (C\omega + \Omega^a)] \sim \dot{\beta}^s \quad (78)$$

is used. Here the tilde outside the square brackets is to be construed to apply to the entire 3 by 1 column matrix within those brackets. With this substitution, the matrix equation corresponding to Eq. (77) becomes\*

$$T^s = I^s [\dot{C}_\omega + \dot{\Omega}^a + \ddot{\beta}^s] + [I^s (\widetilde{C}_\omega) + (\widetilde{C}_\omega) I^s + I^s \widetilde{\Omega}^a + \widetilde{\Omega}^a I^s - (I^s C_\omega + I^s \Omega^a)^\sim] \dot{\beta}^s - I^s \widetilde{\Omega}^a C_\omega + [(\widetilde{C}_\omega) + \widetilde{\Omega}^a] I^s [(C_\omega) + \Omega^a] \quad (79)$$

Again it should be noted that these rotational equations can, with the use of Eqs. (66) and (67), be written in a form not involving  $\Omega^a$ , should this be desirable in a special case.

In most applications, it is convenient to accept Eqs. (79) and (57) as the rotational and translational equations of motion, respectively, of body  $A_s$ , with Eqs. (60), (64), and (65) substituted into Eq. (57) to accommodate vehicle mass-center motion relative to body  $B$ . The result of this substitution is recorded as follows, with the temporary convention that  $\Sigma$  means the sum over  $k$  ranging from 1 to  $n$ :

$$F^s = m^s \{ C \ddot{\Theta} \ddot{X} - (C \ddot{R} + \ddot{r}^s C + C \ddot{e}) \dot{\omega} - \ddot{r}^s \dot{\Omega}^a + \ddot{u}^s - \Sigma_{\mu^k} \ddot{u}^k + C \ddot{e} + 2 [\widetilde{\Omega}^a + (\widetilde{C}_\omega)] \dot{u}^s - 2 C \widetilde{\omega} C^T \Sigma_{\mu^k} \dot{u}^k + 2 C \widetilde{\omega} \dot{e} - 2 \widetilde{\Omega}^a \Sigma_{\mu^k} \dot{u}^k - C \widetilde{\omega} C^T \Sigma_{\mu^k} u^k + [\widetilde{\Omega}^a + (\widetilde{C}_\omega)] u^s - \widetilde{\Omega}^a \Sigma_{\mu^s} u^s - 2 C \widetilde{\omega} C^T \widetilde{\Omega}^a \Sigma_{\mu^k} u^k - \widetilde{\Omega}^a \widetilde{\Omega}^a \Sigma_{\mu^k} u^k + C \widetilde{\omega} \widetilde{\omega} (R + e) - C \widetilde{\omega} \widetilde{\omega} C^T \Sigma_{\mu^k} u^k + [(\widetilde{C}_\omega) (\widetilde{C}_\omega) + 2 (\widetilde{C}_\omega) (\widetilde{\Omega}^a) + \widetilde{\Omega}^a \widetilde{\Omega}^a] (r^s + u^s) \} \quad (80)$$

Again the identity  $(\widetilde{C}_\omega) \equiv C \widetilde{\omega} C^T$  might be substituted, and also the consequence  $(\widetilde{C}_\omega) (\widetilde{C}_\omega) \equiv C \widetilde{\omega} \widetilde{\omega} C^T$ .

Equations (79) and (80) constitute the most general formulation of equations of motion of an appendage sub-body  $A_s$  to be derived in this report. These equations are applicable to a sub-body of an appendage that is undergoing small deformations while rotating in an arbitrary way (as described by  $C$  and  $\Omega^a$ ) relative to a base  $B$  subject to any translation and rotation (as described by  $\ddot{X}$  for the vehicle  $CM$  and  $\Theta$  and  $\omega$  for body  $B$ ). Any number of additional appendages or articulated moving parts may also be attached to  $B$ ; these internal motions will influence  $\Theta$ ,  $\omega$ , and  $e$ .

In practice, the generality of these equations is rarely required. Only in the exceptional case (e.g., the scanning antenna) is there a nonzero  $\Omega^a$ , so  $C$  is usually constant. Then there is no theoretical objection to replacing  $C$  by the identity matrix  $E$ , particularly if there is but one appendage attached to body  $B$ . For reasons not yet apparent, it may be computationally desirable to adopt the view that all bodies  $B$  have only one appendage, even though the appendage may in some cases be composed of several physically distinct structures attached to  $B$ . Then  $e$  is zero, except when a statically unbalanced rotating rigid body is also attached to  $B$ , or a secondary rigid body can translate relative to  $B$ .

For the simplest configuration, with  $C \equiv E$  and  $e \equiv 0$ , the equations of motion (79) and (80) adopt the form

$$T^s = I^s [\dot{\omega} + \ddot{\beta}^s] + [I^s \widetilde{\omega} + \widetilde{\omega} I^s - (\widetilde{I}^s \widetilde{\omega})] \dot{\beta}^s + \widetilde{\omega} I^s \widetilde{\omega} \quad (81)$$

$$F^s = m^s [\ddot{\Theta} \ddot{X} + \ddot{u}^s - \Sigma_{\mu^k} \ddot{u}^k + 2 \widetilde{\omega} (\dot{u}^s - \Sigma_{\mu^k} \dot{u}^k) - (\ddot{R} + \ddot{r}^s) \dot{\omega} + \widetilde{\omega} (u^s - \Sigma_{\mu^k} u^k) + \widetilde{\omega} \widetilde{\omega} (R + r^s + u^s - \Sigma_{\mu^k} u^k)] \quad (82)$$

These equations still permit unrestricted motion of base  $B$ .

The forces  $F^s$  and torques  $T^s$  that appear in the basic differential equations (79) and (80) of sub-body  $A_s$  include terms due to structural interactions with neighboring sub-bodies of the appendage  $A$ . These interactions are represented here as linearly elastic and viscous forces and torques, so they are proportional to the deformations and deformation rates as represented by  $u^1, \dots, u^n, \beta^1, \dots, \beta^n$  and their first derivatives. In practice, viscous damping terms are often unspecified until transformation to modal coordinates has been accomplished.

The resulting approximation is far from perfect in its representation of structural behavior. Even when the materials of a complex structure are essentially linearly elastic, there may be sufficient "play" in the joints of the assembled structure to jeopardize the assumption of linear elasticity of the composite structure. Furthermore, the assumption of viscous damping is not strictly in conformance with the performance of even the simplest structural elements. But for complex structures lacking discrete artificial dampers, the assumptions of linear elasticity and

\*The identity  $(\widetilde{C}_\omega) \equiv C \widetilde{\omega} C^T$  should perhaps be noted, although this substitution into Eq. (79) seems computationally inefficient.

viscous modal damping are almost universal practice in structural dynamics, and these assumptions are reasonably well supported by vibration test data (Refs. 31, 32). They are accordingly the assumptions adopted in this report.

Structural forces and torques in each of the equations typified by Eqs. (79) and (80) therefore may couple these equations to those of every other sub-body of the appendage. The most convenient method of recording all the coupled translational and rotational equations of the  $n$  bodies  $A_1, \dots, A_n$  is with a single matrix equation of dimension  $6n$ .

Because interest is focused presently on the appendage deformations  $u^1, \dots, u^n, \beta^1, \dots, \beta^n$ , only terms in these variables will be written on the left side of the appendage matrix equation. The single  $6n$  by 1 matrix

$$q \equiv [u_1^1 \ u_2^1 \ u_3^1 \ \beta_1^1 \ \beta_2^1 \ \beta_3^1 \ u_1^2 \ \dots \ \beta_3^n]^T \quad (83)$$

fully characterizes the appendage deformations.

Because the equations have been linearized in the appendage deformations, the matrix equation must have the structure

$$M' \ddot{q} + D' \dot{q} + G' \dot{q} + K' q + A' q = L' \quad (84)$$

where by definition  $D'$  and  $K'$  are symmetric and  $G'$  and  $A'$  are skew-symmetric. Inspection of Eqs. (79) and (80) reveals that  $M'$  is a constant symmetric matrix, but  $G'$ ,  $A'$ , and  $K'$  depend on the variables  $\omega$ ,  $\Omega^a$ , and  $C$ . The matrix  $L'$  depends on these variables and in addition on  $\Theta$ ,  $\dot{X}$ ,  $e$ , and the external applied forces that may appear in  $F^s$  and  $T^s$ . The matrix  $D'$  accommodates damping in the structure.

The detailed representation of the  $6n$  by  $6n$  coefficient matrices  $M'$ ,  $G'$ ,  $A'$ , and  $K'$  and the  $6n$  by 1 matrix  $L'$  is facilitated by the introduction of the Boolean operator matrices

$$\left. \begin{aligned} \Sigma_{E0} &= [E \ 0 \ E \ 0 \ \dots \ E \ 0]^T \\ \Sigma_{0E} &= [0 \ E \ 0 \ E \ \dots \ 0 \ E]^T \end{aligned} \right\} \quad (85)$$

where  $E$  and  $0$  are 3 by 3 matrices, the former being the unit matrix and the latter the null matrix. In the present application the matrices  $\Sigma_{E0}$  and  $\Sigma_{0E}$  will generally have  $6n$  rows, but this restriction is not embodied in the defi-

nition, since the matrix dimension is always clear from the context in which it is used.

The operators  $\Sigma_{E0}$  and  $\Sigma_{0E}$  serve to distribute a given 3-row matrix into alternate 3-row partitions of a larger matrix, having perhaps  $6n$  rows. For example, the expression  $\Sigma_{E0} C \Theta \ddot{X}$  is the column matrix

$$[(C \Theta \ddot{X}) \ 0 \ (C \Theta \ddot{X}) \ 0 \ \dots \ (C \Theta \ddot{X}) \ 0]^T$$

which, by inspection of Eqs. (79) and (80), must appear as part of  $L'$  in Eq. (84).

The operators  $\Sigma_{E0}^T$  and  $\Sigma_{0E}^T$  are, on the other hand, effectively selective summing operators, since, for example, when multiplied by a  $6n$ -row matrix they sum respectively the odd- and even-numbered partitioned 3-row submatrices of the  $6n$ -row matrix into a single 3-row matrix. As a specific example,  $\Sigma_{E0}^T \ddot{q}$  provides the 3 by 1 column matrix obtained by adding  $\ddot{u}^1 + \ddot{u}^2 + \dots + \ddot{u}^n$ .

The constant matrix  $M'$  in Eq. (84) is now most easily written in terms of the new symbol  $M$ , defined in terms of 3 by 3 partitioned matrices as

$$M = \begin{bmatrix} m^1 & 0 & 0 & 0 & \dots & 0 \\ 0 & I^1 & 0 & 0 & \dots & 0 \\ 0 & 0 & m^2 & 0 & \dots & 0 \\ 0 & 0 & 0 & I^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & I^n \end{bmatrix} \quad (86)$$

Now  $M'$  can be recorded as

$$M' = M (E - \Sigma_{E0} \Sigma_{E0}^T M / \mathcal{M}) \quad (87)$$

Here  $E$  is an identity matrix of dimension  $6n$  by  $6n$ , and  $\mathcal{M}$  is, as previously, the total vehicle mass (recall  $\mu^s \equiv m^s / \mathcal{M}$ ). The matrix  $M$  may be recognized as the inertia matrix that would be involved in a calculation of cantilever modes of the appendage. This interpretation will be explored in detail in Section III-D on coordinate transformations.

In expansion,  $M'$  appears in terms of 3 by 3 partitioned matrices as shown in Fig. 5. Note that  $M'$  is a symmetric matrix of constants.



$$M' = \begin{bmatrix} m^1 - m^1 m^1 / \mathcal{M} & 0 & -m^1 m^2 / \mathcal{M} & 0 & \dots & 0 \\ 0 & I^1 & 0 & 0 & \dots & 0 \\ -m^2 m^1 / \mathcal{M} & 0 & m^2 - m^2 m^2 / \mathcal{M} & 0 & \dots & 0 \\ 0 & 0 & 0 & I^2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ -m^n m^1 / \mathcal{M} & 0 & -m^n m^2 / \mathcal{M} & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & I^n \end{bmatrix}$$

Fig. 5. Inertia matrix

Before recording the matrix  $G'$  in expanded form, it is useful to examine the coefficient of  $\beta^s$  in Eq. (79) in some detail. It is evident that neither  $\Omega^a$  nor  $C$  changes the character or structure of this coefficient, so the examination is restricted to the special case for which  $C \equiv E$  and  $\Omega^a \equiv 0$ . By expanding the matrices, one finds that

$$I^s \tilde{\omega} + \tilde{\omega} I^s - (\tilde{I^s \omega}) = \begin{bmatrix} 0 & -(I_1 + I_2 - I_3) \omega_3 & (I_1 - I_2 + I_3) \omega_2 \\ (I_1 + I_2 - I_3) \omega_3 & 0 & -(-I_1 + I_2 + I_3) \omega_1 \\ -(I_1 - I_2 + I_3) \omega_2 & (-I_1 + I_2 + I_3) \omega_1 & 0 \end{bmatrix}$$

$$= (\text{tr } I^s) \tilde{\omega} - 2 (\tilde{I^s \omega}) \quad (88)$$

Thus it is evident that the sum of these three matrices is skew-symmetric, although the first two matrices are not. With Eq. (88) substituted into Eq. (79), inspection of this equation and Eq. (80) permits the matrix  $G'$  to be recorded in terms of 3 by 3 partitioned matrices in the expanded form shown in Fig. 6. The substitution  $C \tilde{\omega} C^T = (\tilde{C \omega})$  has also been applied here. Note that all of the  $\dot{q}$  terms on the right side of Eqs. (79) and (80) are accommodated in  $G'$ , leaving only damping terms in the symmetric matrix  $D'$ .

The matrix  $G'$  is skew-symmetric, since the transpose of the matrix in Fig. 6 is also its negative. A more compact and useful representation of this matrix can be obtained by broadening the definition of the operator tilde ( $\sim$ ). As illustrated in Eq. (44), when a tilde appears over a symbol

signifying a 3 by 1 matrix, the corresponding skew-symmetric 3 by 3 matrix is implied. Henceforth, when a tilde appears over any symbol representing a column matrix that can be partitioned into 3 by 1 matrices, a skew-symmetric matrix is implied, which, as a matrix of 3 by 3 partitions, is diagonal, with the 3 by 3 skew-symmetric counterparts of the 3 by 1 elements of the column matrix ranged along the diagonal. For example, on the column matrix

$$\Sigma_{E0} \Omega^a = [\Omega^a \quad 0 \quad \Omega^a \quad 0 \cdots \Omega^a \quad 0]$$

the tilde operator signifies

$$(\Sigma_{E0} \Omega^a)^\sim = \begin{bmatrix} \tilde{\Omega}^a & 0 & 0 & \cdots & \cdot & 0 \\ 0 & 0 & 0 & \cdots & \cdot & 0 \\ 0 & 0 & \tilde{\Omega}^a & \cdots & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \tilde{\Omega}^a & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \quad (89)$$

$G' =$ 

$$\begin{bmatrix}
 2[\tilde{\Omega}^a + (\tilde{C}_\omega)] [m^1 - m^1 m^1 / \mathcal{M}] & 0 & -2[\tilde{\Omega}^a + (\tilde{C}_\omega)] m^1 m^2 / \mathcal{M} & 0 & \cdots & 0 \\
 0 & (\text{tr } I^1) \tilde{\omega} - 2(\tilde{I}^1_\omega) & 0 & 0 & \cdots & 0 \\
 -2[\tilde{\Omega}^a + (\tilde{C}_\omega)] m^2 m^1 / \mathcal{M} & 0 & 2[\tilde{\Omega}^a + (\tilde{C}_\omega)] [m^2 - m^2 m^2 / \mathcal{M}] & 0 & \cdots & 0 \\
 0 & 0 & 0 & (\text{tr } I^2) \tilde{\omega} - 2(\tilde{I}^2_\omega) & \cdots & 0 \\
 \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 -2[\tilde{\Omega}^a + (\tilde{C}_\omega)] m^a m^1 / \mathcal{M} & 0 & -2[\tilde{\Omega}^a + (\tilde{C}_\omega)] m^a m^2 / \mathcal{M} & 0 & \cdots & 0 \\
 0 & 0 & 0 & 0 & \cdots & (\text{tr } I^a) \tilde{\omega} - 2(\tilde{I}^a_\omega)
 \end{bmatrix}$$

Fig. 6. Coriolis matrix

With this added convention, after inspection of Fig. 6 and Eqs. (79) and (80), the matrix  $G'$  may be written in the compact form

$$G' = 2M \{ (\Sigma_{E0} \Omega^a)^\sim + (\Sigma_{E0} C\omega)^\sim - \Sigma_{E0} [(\tilde{C}\omega) + \tilde{\Omega}^a] \Sigma_{E0}^T M / \mathcal{M} \} + M [(\Sigma_{0E} \Omega^a)^\sim + (\Sigma_{0E} C\omega)^\sim] \\ + [(\Sigma_{0E} \Omega^a)^\sim + (\Sigma_{0E} C\omega)^\sim] M - [M (\Sigma_{0E} \Omega^a + \Sigma_{0E} C\omega)]^\sim \quad (90)$$

The matrices  $A'$  and  $K'$  appear in compact form as

$$\left. \begin{aligned} A' &= M \{ [(\Sigma_{E0} \dot{\Omega}^a)^\sim + (\Sigma_{E0} C\dot{\omega})^\sim] - \Sigma_{E0} [\tilde{\Omega}^a + (C\tilde{\omega})] \Sigma_{E0}^T M / \mathcal{M} \} \\ K' &= M \{ (\Sigma_{E0} \Omega^a)^\sim (\Sigma_{E0} \Omega^a)^\sim + 2 (\Sigma_{E0} C\omega)^\sim (\Sigma_{E0} \Omega^a)^\sim + (\Sigma_{E0} C\omega)^\sim (\Sigma_{E0} C\omega)^\sim \\ &\quad - \Sigma_{E0} [\tilde{\Omega}^a \tilde{\Omega}^a + 2 (\tilde{C}\omega) \tilde{\Omega}^a + (\tilde{C}\omega) (\tilde{C}\omega)] \Sigma_{E0}^T M / \mathcal{M} \} + K \end{aligned} \right\} \quad (91)$$

where  $K$  is the stiffness matrix that defines the structural interaction forces and torques induced by deformation of the appendage. The stiffness matrix  $K$  is symmetric (see Refs. 11-13), but in general there does exist a skew-symmetric matrix  $A'$ . Specifically, the terms in Eq. (91) involving angular accelerations  $\dot{\Omega}^a$  and  $\dot{\omega}$  are skew-symmetric, and all other terms are symmetric.

Figure 7 shows the first few terms in the upper left corner of the matrix  $A' + K' - K$ . It is important to recognize that in checking this matrix for symmetry one must, in transposing the partitioned matrix, remember to transpose the 3 by 3 matrices within the partitions. Since the transpose of a skew-symmetric matrix is its negative, one has, for example,

$$(\tilde{\Omega}^a)^T = -\tilde{\Omega}^a$$

and

$$[\tilde{\Omega}^a \tilde{\Omega}^a]^T = (\tilde{\Omega}^a)^T (\tilde{\Omega}^a)^T = (-\tilde{\Omega}^a) (-\tilde{\Omega}^a) = \tilde{\Omega}^a \tilde{\Omega}^a$$

This permits the previously noted identification of the angular acceleration terms as the skew-symmetric terms that compose  $A'$ .

There remains only the explicit identification of the elements in the matrix  $L'$  of Eq. (84). Compact representation is accomplished here with the new definition

$$r \equiv [r^1 \ 0 \ r^2 \ 0 \ \cdots \ r^n \ 0]^T \quad (92)$$

The general interpretation of the tilde operator now provides

$$\tilde{r} \equiv \begin{bmatrix} \tilde{r}^1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \tilde{r}^2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & \tilde{r}^n & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \quad (93)$$

Now the  $6n$  by 1 matrix  $L'$  can be written as

$$L' = -M \Sigma_{0E} [\dot{\Omega}^a + C\dot{\omega} - \tilde{\Omega}^a C\omega] - [\Sigma_{0E} (\Omega^a + C\omega)]^\sim M \Sigma_{0E} (\Omega^a + C\omega) \\ - M \Sigma_{E0} C [\odot \ddot{X} - (\tilde{R} + \tilde{e}) \dot{\omega} + \ddot{e} - 2\tilde{e}\dot{\omega} + \tilde{\omega}\tilde{\omega}(R + e)] \\ + M \{ \tilde{r} \Sigma_{E0} (\dot{\Omega}^a + C\dot{\omega}) - [(\Sigma_{E0} \Omega^a)^\sim (\Sigma_{E0} \Omega^a)^\sim + 2 (\Sigma_{E0} C\omega)^\sim (\Sigma_{E0} \Omega^a)^\sim + (\Sigma_{E0} C\omega)^\sim (\Sigma_{E0} C\omega)^\sim] r \} + \lambda \quad (94)$$

where the new symbol  $\lambda$  is a column matrix in which to store any forces or torques applied to the sub-bodies  $A_1, \dots, A_n$  other than the structural interaction forces induced by deformations. For example, field forces such as gravity forces on the appendage belong in  $\lambda$ , and any attitude-control jets on the appendage would contribute to  $\lambda$  in a manner established by a control law.

In expanded form, the matrix  $L'$  appears as a column of 3 by 1 submatrices in Fig. 8, in which the forces and torques on body  $j$  in  $\lambda$  are represented respectively by  $f^j$  and  $\tau^j$ .

This concludes the derivation of the equations of motion of an elastic appendage on a base undergoing arbitrary

motion. The final result is Eq. (84), with matrices  $q$ ,  $M'$ ,  $G'$ ,  $K'$ ,  $A'$ , and  $L'$  given in compact form by Eqs. (83), (87), (90), (91), and (94). These equations are complicated by the generality introduced with time-varying  $C$ , which permits application to a flexible appendage of changing attitude relative to its base (e.g., a scanning antenna). Additional complexity stems from the variable  $e$ , which is present to accommodate movable parts within the vehicle (other than the appendage). Specialization to the simplest configuration, with  $C \equiv E$  and  $e \equiv 0$ , led to the basic Eqs. (81) and (82) for the typical submass  $A_n$ .

When these equations are combined as a single matrix equation such as Eq. (84), one obtains the simpler result

$$\begin{aligned} M(E - \Sigma_{E0} \Sigma_{E0}^T M / \eta) \ddot{q} + \{2M[(\Sigma_{E0} \dot{\omega})^\sim - \Sigma_{E0} \tilde{\omega} \Sigma_{E0}^T M / \eta] + M(\Sigma_{0E} \dot{\omega})^\sim + (\Sigma_{0E} \dot{\omega})^\sim M - [M \Sigma_{0E} \dot{\omega}]^\sim + D'\} \dot{q} \\ + \{M[(\Sigma_{E0} \dot{\omega})^\sim - \Sigma_{E0} \tilde{\omega} \Sigma_{E0}^T M / \eta] + M[(\Sigma_{E0} \dot{\omega})^\sim (\Sigma_{E0} \dot{\omega})^\sim - \Sigma_{E0} \tilde{\omega} \tilde{\omega} \Sigma_{E0}^T M / \eta] + K\} q = \\ - M \Sigma_{0E} \dot{\omega} - (\Sigma_{0E} \dot{\omega})^\sim M (\Sigma_{0E} \dot{\omega}) - M \{\Sigma_{E0} [\dot{\omega} \ddot{X} - \tilde{R} \dot{\omega} + \tilde{\omega} \tilde{\omega} R] - \tilde{r} \Sigma_{E0} \dot{\omega} - (\Sigma_{E0} \dot{\omega})^\sim (\Sigma_{E0} \dot{\omega})^\sim r\} + \lambda \end{aligned} \quad (95)$$

Even in this case the equations remain complex in appearance. Although the specialization has greatly reduced the number of terms in the equation, no simplification of the basic structure of the equations has resulted. For the special configuration as well as the more general one, the equations of motion have the structure of Eq. (84). In both cases  $M'$  is constant and symmetric;  $G'$  is skew-symmetric and depending linearly on the variable and generally unknown angular velocity  $\omega$  of the base  $B$ ;  $K'$  is the sum of a constant symmetric stiffness matrix  $K$  and a variable symmetric matrix depending on second-degree terms in  $\omega$ ;  $A'$  is a variable skew-symmetric matrix depending linearly on  $\dot{\omega}$ ; and  $L'$  is composed of applied forces and torques, nonlinear functions of  $\omega$ , linear terms in  $\dot{\omega}$ , and the unknown mass-center motion  $\ddot{X}$ .

It may appear that the various matrix operators introduced here serve to obfuscate rather than clarify results, and that the appendage equations are too hopelessly large and complex for efficient simulation. Both of these impressions would be correct if the derivation were to end at this point. Many of the matrices appearing here are quite

sparse, and their use in a digital computer program is rendered inefficient by the number of multiplications by zero. The number of sub-bodies  $A_1, \dots, A_n$  required for an acceptable simulation of a flexible appendage may be quite large (30 is not an unreasonably large value for  $n$ ), and the matrix Eq. (84) is equivalent to  $6n$  simultaneous coupled scalar differential equations.

Practical utility can be demonstrated for these equations only after transformation to carefully selected coordinates permits the exercise of judgment in ignoring all but a small number of variables. It is remarkable that in introducing a coordinate transformation and combining the resulting equations with appropriate equations for the balance of the vehicle, seemingly complex matrix combinations will emerge with simple physical interpretations, and coordinate truncation will permit meaningful preliminary designs of vehicle attitude-control systems to be accomplished without recourse to the computer. The results of the coordinate transformations, however, will be deferred until Section III-D so that we may first consider the remainder of the equations of motion of the vehicle.

$$A' + K' - K =$$

$$\begin{bmatrix} [\tilde{\Omega}^a + (\tilde{C}_\omega) + \tilde{\Omega}^a \tilde{\Omega}^a + 2(\tilde{C}_\omega) \tilde{\Omega}^a + (\tilde{C}_\omega) (\tilde{C}_\omega)] [m^1 - m^1 m^1 / \eta] & 0 & -[\tilde{\Omega}^a + (\tilde{C}_\omega) + \tilde{\Omega}^a \tilde{\Omega}^a + 2(\tilde{C}_\omega) \tilde{\Omega}^a + (\tilde{C}_\omega) (\tilde{C}_\omega)] m^2 m^1 / \eta & \dots \\ 0 & 0 & 0 & \dots \\ -[\tilde{\Omega}^a + (\tilde{C}_\omega) + \tilde{\Omega}^a \tilde{\Omega}^a + 2(\tilde{C}_\omega) \tilde{\Omega}^a + (\tilde{C}_\omega) (\tilde{C}_\omega)] m^1 m^2 / \eta & 0 & [\tilde{\Omega}^a + (\tilde{C}_\omega) + \tilde{\Omega}^a \tilde{\Omega}^a + 2(\tilde{C}_\omega) \tilde{\Omega}^a + (\tilde{C}_\omega) (\tilde{C}_\omega)] [m^2 - m^2 m^2 / \eta] & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \dots \\ \vdots & \vdots & \vdots & \dots \end{bmatrix}$$

Fig. 7. Modified stiffness matrix

$$L' = \begin{bmatrix} -m^1 \{C [\Theta \ddot{X} - (\tilde{R} + \tilde{e}) \dot{\omega} + \ddot{e} - 2\tilde{e}\dot{\omega} + \tilde{\omega}\tilde{\omega} (R + e) - \tilde{r}^1 (\dot{\Omega}^a + C\dot{\omega})] + [\tilde{\Omega}^a \tilde{\Omega}^a + 2(\tilde{C}_\omega) \tilde{\Omega}^a + (\tilde{C}_\omega) (\tilde{C}_\omega)] r^1\} + f^1 \\ -I^1 [\dot{\Omega}^a + C\dot{\omega} - \tilde{\Omega}^a C\omega] - [\tilde{\Omega}^a + (\tilde{C}_\omega)] I^1 [\Omega^a + C\omega] + l^1 \\ -m^2 \{C [\Theta \ddot{X} - (\tilde{R} + \tilde{e}) \dot{\omega} + \ddot{e} - 2\tilde{e}\dot{\omega} + \tilde{\omega}\tilde{\omega} (R + e) - \tilde{r}^2 (\dot{\Omega}^a + C\dot{\omega})] + [\tilde{\Omega}^a \tilde{\Omega}^a + 2(\tilde{C}_\omega) \tilde{\Omega}^a + (\tilde{C}_\omega) (\tilde{C}_\omega)] r^2\} + f^2 \\ -I^2 [\dot{\Omega}^a + C\dot{\omega} - \tilde{\Omega}^a C\omega] - [\tilde{\Omega}^a + (\tilde{C}_\omega)] I^2 [\Omega^a + C\omega] + l^2 \\ \vdots \\ \vdots \\ -I^n [\dot{\Omega}^a + C\dot{\omega} - \tilde{\Omega}^a C\omega] - [\tilde{\Omega}^a + (\tilde{C}_\omega)] I^n [\Omega^a + C\omega] + l^n \end{bmatrix}$$

Fig. 8. Forcing function

### C. Vehicle Equations

Immediately upon shifting attention from the appendage deformations to the time behavior of the total vehicle, the analyst confronts a basic question. Should he simply isolate the rigid body to which the appendage is attached and derive its equations of motion, treating the forces and torques applied to it by the appendage as "external"? Or should he instead write equations of motion for a dynamic system that includes both the rigid body and the flexible appendage? A parallel question was considered in Section II, in which augmented-body methods were contrasted with nested-body methods. In the context of hybrid-coordinate analysis, the answer to this question is complicated by the anticipated truncation of modal coordinates of the appendage. If the analyst isolates the body to which the appendage is attached, he must be sure that in the truncation process he retains a valid representation of the interaction forces between base and appendage. Because of this difficulty, it seems generally desirable to include the appendage with the rigid body in the dynamic system, and this will be the method used here. In certain cases, however, when the vehicle undergoes large changes in configuration, it may be most efficient computationally to treat the appendage interaction forces as external forces applied to the rigid body. Then some artifice is required (such as the "synthetic modes" of Ref. 28) to assure that coordinate truncation does not invalidate the interaction force expression. This method is explored in Section III-E of this report.

The immediate task is the derivation of equations of motion of a typical vehicle that includes flexible appendages. No attempt will be made here to establish a general set of equations applicable to virtually all space vehicles. Attention is instead restricted to a class of vehicle that may be modeled as a rigid body  $B_1$  with two flexible appendages ( $A^1$  and  $A^2$ ), a balanced rigid symmetric rotor  $B_3$ , and a single rigid body  $B_2$  attached to  $B_1$  with a single degree of translational freedom.

The vehicle sketched in Fig. 3a does not quite meet the restrictions of the vehicle equations to be derived here; the rotor  $B_1$  of Fig. 3a is not rigid and symmetric, since it includes an internal body  $B_1$  and two flexible appendages  $A^1$  and  $A^2$ . Figure 9 better typifies the vehicle to which the following equations apply directly. It is an idealized tri-spin satellite, designed to maintain spin stabilization with the rigid symmetric rotor  $B_3$ , while a control system between bodies  $B_1$  and  $B_3$  maintains the rigid platform  $B_1$  in an earth-pointing attitude. The flexible antenna  $A^1$  is attached to  $B_1$  at a hinge that permits limited controlled relative motion. A solar-cell array  $A^2$

is also attached to  $B_1$ , with an auxiliary control system maintaining the required sun-pointing orientation. A linear oscillator  $B_2$  acts as a nutation damper for the satellite; its single degree of translational freedom is in a direction transverse to the bearing axis between  $B_1$  and  $B_3$ .

Although the vehicle of Fig. 9 is of substantial interest in itself—and by specializing to omit components or fix joints one can use its equations of motion for a very wide range of space vehicles—its derivation is included here principally for its illustrative value. The analyst must be expected to approach the derivation of equations of motion of each space vehicle configuration individually, if computational efficiency is to result. The appendage equations derived in Section III-B, on the other hand, should be applied directly to any flexible appendage within the definition offered, and should not require rederivation for specific cases.

One may write in application to any material system the basic Newton-Euler equations,  $\mathbf{F} = m\mathbf{a}$  and  $\mathbf{T} = \dot{\mathbf{H}}$ , where  $m$  is the vehicle mass,  $\mathbf{F}$  is the resultant applied force, and  $\mathbf{a}$ ,  $\mathbf{H}$ , and  $\mathbf{T}$  are, respectively, the inertial acceleration, angular momentum, and resultant torque, all referred to the vehicle mass center.

In deriving the rotational equation from

$$\mathbf{T} = \dot{\mathbf{H}} \quad (96)$$

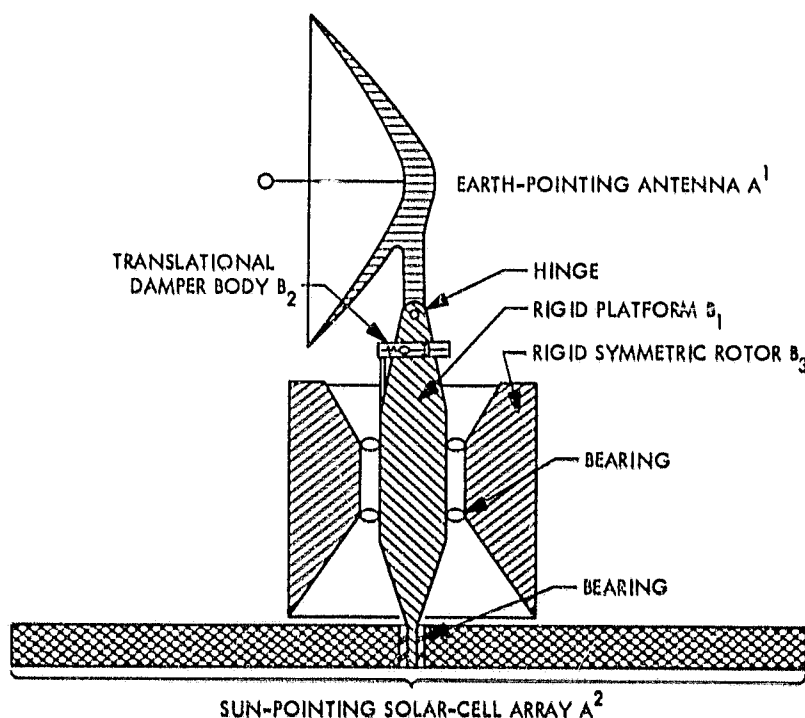


Fig. 9. Tri-spin satellite (cross-sectional view)

one simply substitutes the angular momentum definition

$$\mathbf{H} \equiv \int \mathbf{p} \times \dot{\mathbf{p}} dm \quad (97)$$

where  $\mathbf{p}$  is the generic position vector from the vehicle mass center to a typical differential mass element of the vehicle.

Define a new vector  $\mathbf{p}$  as the generic position vector from the point  $O$  fixed in body  $B_1$ , and replace  $\mathbf{p}$  in Eq. (91) by  $\mathbf{p} + \mathbf{c}$  (recalling that  $-\mathbf{c}$  defines the displacement in  $B_1$  of the vehicle mass center  $CM$  from its nominal location at  $O$ ). The result is then differentiated (making use once more of Eq. 38) and the integral rewritten as follows, employing a convention whereby differentiation in the reference frame of  $B_1$  is signified by an overcircle (e.g.,  $\dot{\mathbf{p}}$ ):

$$\begin{aligned} \mathbf{H} &= \int (\mathbf{p} + \mathbf{c}) \times (\dot{\mathbf{p}} + \dot{\mathbf{c}}) dm \\ &= \int (\mathbf{p} + \mathbf{c}) \times (\dot{\mathbf{p}} + \dot{\mathbf{c}}) dm \\ &\quad + \int (\mathbf{p} + \mathbf{c}) \times [\boldsymbol{\omega} \times (\mathbf{p} + \mathbf{c})] dm \\ &= \mathbf{c} \times \int (\dot{\mathbf{p}} + \dot{\mathbf{c}}) dm + \int \mathbf{p} \times \dot{\mathbf{p}} dm - \dot{\mathbf{c}} \times \int \mathbf{p} dm \\ &\quad + \int \mathbf{p} dm \times (\boldsymbol{\omega} \times \mathbf{c}) \\ &\quad + \int \mathbf{p} \times (\boldsymbol{\omega} \times \mathbf{p}) dm + \mathbf{c} \times \left[ \boldsymbol{\omega} \times \int (\mathbf{p} + \mathbf{c}) dm \right] \end{aligned} \quad (98)$$

The mass-center definition requires

$$\int (\mathbf{p} + \mathbf{c}) dm = 0 \quad (99)$$

so

$$\int \mathbf{p} dm = -\mathbf{c} \int dm = -\mathbf{c}M \quad (100)$$

and

$$\int (\dot{\mathbf{p}} + \dot{\mathbf{c}}) dm \equiv \frac{d}{dt} \int (\mathbf{p} + \mathbf{c}) dm = 0 \quad (101)$$

Thus many of the integrals in Eq. (98) become either zero or a simple vector. The immediate result is

$$\begin{aligned} \mathbf{H} &= \int \mathbf{p} \times \dot{\mathbf{p}} dm + \dot{\mathbf{c}} \times M\mathbf{c} - M\mathbf{c} \times (\boldsymbol{\omega} \times \mathbf{c}) \\ &\quad + \int \mathbf{p} \times (\boldsymbol{\omega} \times \mathbf{p}) dm \\ &= \int \mathbf{p} \times \dot{\mathbf{p}} dm + M\dot{\mathbf{c}} \times \mathbf{c} + \int \mathbf{p} \times (\boldsymbol{\omega} \times \mathbf{p}) dm \end{aligned} \quad (102)$$

The last integral in Eq. (102) may be recognized as the dot-product of the inertia dyadic  $\mathbf{I}$  of the vehicle for point  $O$  and the inertial angular velocity of the base  $B_1$ . This identification can be established in detail as follows:

$$\begin{aligned} \int \mathbf{p} \times (\boldsymbol{\omega} \times \mathbf{p}) dm &= \int [\mathbf{p} \cdot \mathbf{p} \boldsymbol{\omega} - \mathbf{p} \cdot \boldsymbol{\omega} \mathbf{p}] dm \\ &= \int [\mathbf{p} \cdot \mathbf{p} \mathbf{E} - \mathbf{p} \mathbf{p}] dm \cdot \boldsymbol{\omega} \equiv \mathbf{I} \cdot \boldsymbol{\omega} \end{aligned} \quad (103)$$

where  $\mathbf{E}$  is the unit dyadic (so  $\mathbf{E} \cdot \boldsymbol{\omega} = \boldsymbol{\omega}$ ). Thus the vehicle angular momentum with respect to its mass center is

$$\mathbf{H} = \mathbf{I} \cdot \boldsymbol{\omega} + M\dot{\mathbf{c}} \times \mathbf{c} + \int \mathbf{p} \times \dot{\mathbf{p}} dm \quad (104)$$

It should be noted that when  $\mathbf{c}$  is due only to appendage deformations, assumed arbitrarily small, the product  $M\dot{\mathbf{c}} \times \mathbf{c}$  is of second degree and therefore negligible. For the vehicle illustrated in Fig. 9, the mass center of the appendage  $A^2$  is nominally fixed in  $B_1$ , so the large relative motion permitted between these components does not contribute to  $\mathbf{c}$ . The mass of the translating body  $B_2$  makes the contributions of this damper body to  $\mathbf{c}$  small in an engineering sense (relatively small, but not infinitesimal), and for certain purposes (e.g., stability analysis of balanced tri-spin motion) the damper translation may be assumed arbitrarily small. Thus the contribution of  $B_2$  to  $M\dot{\mathbf{c}} \times \mathbf{c}$  may also be negligible. For the vehicle illustrated, only the rotation of  $A^1$  on its hinge might contribute materially to  $\mathbf{c}$ , and even this might be only an occasional influence. The term  $M\dot{\mathbf{c}} \times \mathbf{c}$  is preserved in this derivation for generality, but it should be recognized that it should be retained only in exceptional applications.

The rotational equation (96) requires the inertial differentiation of the angular momentum (Eq. 104), yielding



(with dyadic differentiation as in Eq. 70)

$$\begin{aligned} \mathbf{T} &= \mathbf{I} \cdot \dot{\boldsymbol{\omega}} + \dot{\mathbf{I}} \cdot \boldsymbol{\omega} + m\ddot{\mathbf{c}} \times \mathbf{c} + \frac{d}{dt} \int \mathbf{p} \times \dot{\mathbf{p}} dm \\ &= \mathbf{I} \cdot \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I} \cdot \boldsymbol{\omega} + \dot{\mathbf{I}} \cdot \boldsymbol{\omega} + m[\ddot{\mathbf{c}} + 2\boldsymbol{\omega} \times \dot{\mathbf{c}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{c}) + \dot{\boldsymbol{\omega}} \times \mathbf{c}] \times \mathbf{c} + \frac{d}{dt} \int \mathbf{p} \times \dot{\mathbf{p}} dm \end{aligned} \quad (105)$$

Note that Euler's equations emerge here for the special case of a rigid vehicle, since then only the first two terms on the right are nonzero. Another special case of interest is that of the nominally stationary base  $B_1$ , for which  $\boldsymbol{\omega}$  is infinitesimal and Eq. (105) linearizes to

$$\mathbf{T} = \mathbf{I} \cdot \dot{\boldsymbol{\omega}} + \int \mathbf{p} \times \ddot{\mathbf{p}} dm \quad (106)$$

The challenge of Eqs. (105) and (106) is the replacement of the integrals by more convenient functions. Relative derivatives  $\dot{\mathbf{p}}$  and  $\ddot{\mathbf{p}}$  will exist only for mass elements that move relative to rigid body  $B_1$ , so the separate contributions of the rotor  $B_3$ , the damper  $B_2$ , and the flexible appendages  $A^1$  and  $A^2$  can be calculated in turn.

For the rotor, the relevant integral is simply

$$\frac{d}{dt} \int_{B_3} \mathbf{p} \times \dot{\mathbf{p}} dm = \frac{d}{dt} \int_{B_3} \mathbf{p} \times (\boldsymbol{\Omega} \times \mathbf{p}) dm = \frac{d}{dt} (\mathbf{J} \cdot \boldsymbol{\Omega}) = \dot{\mathbf{h}} \quad (107)$$

where  $\boldsymbol{\Omega}$  is the angular velocity of  $B_3$  relative to  $B_1$ ,  $\mathbf{J}$  is the inertia dyadic of the rotor, and  $\mathbf{h}$  is the angular momentum of the rotor  $B_3$  relative to the base  $B_1$ .

For the translating body  $B_2$ , the integration is trivial, since all parts of  $B_2$  have the same velocity  $\dot{\mathbf{p}}$  in  $B_1$ . Thus if  $m$  is the mass of  $B_2$ , and its position relative to  $O$  is given in the  $B_1$ -fixed vector basis  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ , by  $b\mathbf{b}_3 + \xi\mathbf{b}_1$ ,

with  $b$  constant and  $\xi$  the single translation variable, the integral becomes

$$\frac{d}{dt} \int_{B_2} \mathbf{p} \times \dot{\mathbf{p}} dm = \frac{d}{dt} (mb\xi\dot{\mathbf{b}}_2) = mb\ddot{\xi}\mathbf{b}_2 + mb\xi\dot{\boldsymbol{\omega}} \times \mathbf{b}_2 \quad (108)$$

Finally, the contributions of the appendages to this integral must be sought. Return for this purpose to the concept of the discrete parameter model of the appendage, and the definitions of  $\mathbf{R}, \mathbf{r}^s$ , and  $\mathbf{u}^s$  in Eq. (36) and Fig. 4. In addition, let  $\boldsymbol{\rho}^s$  represent a generic position vector from the mass center  $P_s$  of sub-body  $A_s$  to a typical differential mass element in  $A_s$ . The appropriate integral then becomes, for a typical appendage  $A$  (here either  $A^1$  or  $A^2$ ),

$$\int_A \mathbf{p} \times \dot{\mathbf{p}} dm = \int_A (\mathbf{R} + \mathbf{r}^s + \mathbf{u}^s + \boldsymbol{\rho}^s) \times (\dot{\mathbf{r}}^s + \dot{\mathbf{u}}^s + \dot{\boldsymbol{\rho}}^s) dm \quad (109)$$

Yet another application of the vector differentiation formula in Eq. (38) is required for the interpretation of Eq. (109), and this can be done conveniently by writing the individual deformation vectors in terms of appropriate vector arrays (as in Eqs. 37 and 72). Recalling that  $\boldsymbol{\Omega}^a$  is the angular velocity of  $\{\mathbf{a}\}$  relative to  $\{\mathbf{b}\}$  for a given appendage  $A$ , and  $\{\mathbf{a}\}^T \dot{\boldsymbol{\rho}}^s$  is the angular velocity of  $A_s$  relative to  $A$ , one can write Eq. (109) as

$$\begin{aligned} \int_A \mathbf{p} \times \dot{\mathbf{p}} dm &= \int_A (\mathbf{R} + \mathbf{r}^s + \mathbf{u}^s + \boldsymbol{\rho}^s) \times [\boldsymbol{\Omega}^a \times (\mathbf{r}^s + \mathbf{u}^s) + \{\mathbf{a}\}^T \dot{\mathbf{u}}^s + (\boldsymbol{\Omega}^a + \{\mathbf{a}\}^T \dot{\boldsymbol{\rho}}^s) \times \boldsymbol{\rho}^s] dm \\ &= \mathbf{R} \times \{\mathbf{a}\}^T \int_A \dot{\mathbf{u}}^s dm + \int_A \mathbf{r}^s \times \{\mathbf{a}\}^T \dot{\mathbf{u}}^s dm + \mathbf{R} \times \left[ \boldsymbol{\Omega}^a \times \int_A (\mathbf{r}^s + \mathbf{u}^s) dm \right] \\ &\quad + \int_A \mathbf{r}^s \times [\boldsymbol{\Omega}^a \times (\mathbf{r}^s + \mathbf{u}^s)] dm + \int_A \mathbf{u}^s \times (\boldsymbol{\Omega}^a \times \mathbf{r}^s) dm + \int_A \boldsymbol{\rho}^s \times [(\boldsymbol{\Omega}^a + \{\mathbf{a}\}^T \dot{\boldsymbol{\rho}}^s) \times \boldsymbol{\rho}^s] dm \end{aligned} \quad (110)$$

since in every term the integral over  $A$  can be replaced by the sum of  $n$  integrals over the individual sub-bodies  $A_1, \dots, A_n$ , and

$$\int_{A_1} \boldsymbol{\rho}^s = 0$$

(Remember that  $\mathbf{u}^s$  and  $\{\mathbf{a}\}^T \dot{\beta}^s$  are infinitesimal.) As for the rotor equation (107), the last term in Eq. (110) can be written in terms of the inertia dyadics  $\mathbf{I}^s$  of the sub-bodies. Every other integral in Eq. (110) permits factorization of

$$\int_{A_s} dm = m_s$$

Thus that equation becomes (with  $\Sigma$  denoting the sum over  $s$  ranging from 1 to  $n$ )

$$\begin{aligned} \int_A \mathbf{p} \times \dot{\mathbf{p}} dm &= \mathbf{R} \times \{\mathbf{a}\}^T \Sigma m_s \dot{\mathbf{u}}^s + \Sigma \mathbf{r}^s \times \{\mathbf{a}\}^T m_s \dot{\mathbf{u}}^s \\ &+ \mathbf{R} \times [\Omega^a \times \Sigma m_s (\mathbf{r}^s + \mathbf{u}^s)] + \Sigma m_s \mathbf{r}^s \times [\Omega^a \times (\mathbf{r}^s + \mathbf{u}^s)] + \Sigma m_s \mathbf{u}^s \times (\Omega^a \times \mathbf{r}^s) + \Sigma \mathbf{I}^s \cdot \{\mathbf{a}\}^T \dot{\beta}^s \end{aligned} \quad (111)$$

Equation (105) requires the inertial derivative of Eq. (111), which is

$$\begin{aligned} \frac{d}{dt} \int_A \mathbf{p} \times \dot{\mathbf{p}} dm &= \omega \times [\mathbf{R} \times \{\mathbf{a}\}^T \Sigma m_s \dot{\mathbf{u}}^s] + \mathbf{R} \times [\{\mathbf{a}\}^T \Sigma m_s \ddot{\mathbf{u}}^s + (\omega + \Omega^a) \times \{\mathbf{a}\}^T \Sigma m_s \dot{\mathbf{u}}^s] \\ &+ (\omega + \Omega^a) \times \Sigma \mathbf{r}^s \times \{\mathbf{a}\}^T m_s \dot{\mathbf{u}}^s + \Sigma \mathbf{r}^s \times [\{\mathbf{a}\}^T m_s \ddot{\mathbf{u}}^s + (\omega + \Omega^a) \times \{\mathbf{a}\}^T m_s \dot{\mathbf{u}}^s] \\ &+ (\omega \times \mathbf{R}) \times [\Omega^a \times \Sigma m_s (\mathbf{r}^s + \mathbf{u}^s)] + \mathbf{R} \times [(\dot{\Omega}^a + \omega \times \Omega^a) \times \Sigma m_s (\mathbf{r}^s + \mathbf{u}^s)] \\ &+ \mathbf{R} \times [\Omega^a \times \Sigma m_s \{\mathbf{a}\}^T \dot{\mathbf{u}}^s] + \mathbf{R} \times \{\Omega^a \times \Sigma m_s [(\omega + \Omega^a) \times (\mathbf{r}^s + \mathbf{u}^s)]\} \\ &+ \Sigma m_s [(\omega \times \Omega^a) \times \mathbf{r}^s] \times [\Omega^a \times (\mathbf{r}^s + \mathbf{u}^s)] + \Sigma m_s \mathbf{r}^s \times [(\dot{\Omega}^a + \omega \times \Omega^a) \times (\mathbf{r}^s + \mathbf{u}^s)] \\ &+ \Sigma m_s \mathbf{r}^s \times \{\Omega^a \times [(\omega + \Omega^a) \times (\mathbf{r}^s + \mathbf{u}^s)]\} \\ &+ \Sigma m_s \{\mathbf{a}\}^T \dot{\mathbf{u}}^s \times (\Omega^a \times \mathbf{r}^s) + \Sigma m_s [(\omega + \Omega^a) \times \mathbf{u}^s] \times (\Omega^a \times \mathbf{r}^s) \\ &+ \Sigma m_s \mathbf{u}^s \times [(\dot{\Omega}^a + \omega \times \Omega^a) \times \mathbf{r}^s] + \Sigma m_s \mathbf{u}^s \times \{\Omega^a \times [(\omega \times \Omega^a) \times \mathbf{r}^s]\} \\ &+ \Sigma \mathbf{I}^s \cdot [\{\mathbf{a}\}^T \ddot{\beta}^s + (\omega + \Omega^a) \times \{\mathbf{a}\}^T \dot{\beta}^s] + (\omega + \Omega^a) \times \mathbf{I}^s \cdot \{\mathbf{a}\}^T \dot{\beta}^s - \mathbf{I}^s \times (\omega + \Omega^a) \cdot \{\mathbf{a}\}^T \dot{\beta}^s \end{aligned} \quad (112)$$

This unwieldy expression, repeated for each appendage attached to body  $B_1$ , must together with Eqs. (107) and (108) be substituted into the rotational equation of motion (Eq. 105) in the general case of arbitrary base motion. Furthermore, the contribution of appendage deformation to  $\mathbf{I}$  in Eq. (105) would have to be calculated. In most engineering applications, this degree of complexity is unwarranted. In the foregoing, only appendage deformations have been restricted, and these have been assumed to remain infinitesimal. In fact, they are at best small in an engineering sense only, and deformation velocities and accelerations probably exceed, in most space vehicles, those due to prescribed relative motions (such as  $\Omega^a$  and  $\dot{\Omega}^a$ ). It is thus most reasonable in an engineering application to linearize in these latter variables also, although it should be understood that the resulting equations lack the rigor that can be claimed (however artificially) for equations such as Eq. (105), linearized in deformations only. With linearization in  $\Omega^a$  and  $\dot{\Omega}^a$ , Eq. (112) becomes

$$\begin{aligned} \frac{d}{dt} \int_A \mathbf{p} \times \dot{\mathbf{p}} dm &= \omega \times [\mathbf{R} \times \{\mathbf{a}\}^T \Sigma m_s \dot{\mathbf{u}}^s] + \mathbf{R} \times [\{\mathbf{a}\}^T \Sigma m_s \ddot{\mathbf{u}}^s + \omega \times \{\mathbf{a}\}^T \Sigma m_s \dot{\mathbf{u}}^s] \\ &+ \omega \times (\Sigma \mathbf{r}^s \times \{\mathbf{a}\}^T m_s \dot{\mathbf{u}}^s) + \Sigma \mathbf{r}^s \times [\{\mathbf{a}\}^T m_s \ddot{\mathbf{u}}^s + \omega \times \{\mathbf{a}\}^T m_s \dot{\mathbf{u}}^s] \\ &+ (\omega \times \mathbf{R}) \times (\Omega^a \times \Sigma m_s \mathbf{r}^s) + \mathbf{R} \times (\dot{\Omega}^a + \omega \times \Omega^a) \times \Sigma m_s \mathbf{r}^s + \mathbf{R} \times [\Omega^a \times (\omega \times \mathbf{r}^s)] \\ &+ \Sigma m_s \mathbf{r}^s \times [(\dot{\Omega}^a + \omega \times \Omega^a) \times \mathbf{r}^s] + \Sigma m_s \mathbf{r}^s \times [\Omega^a \times (\omega \times \mathbf{r}^s)] \\ &+ \Sigma \mathbf{I}^s \cdot [\{\mathbf{a}\}^T \ddot{\beta}^s + \omega \times \{\mathbf{a}\}^T \dot{\beta}^s] + \omega \times \mathbf{I}^s \cdot \{\mathbf{a}\}^T \dot{\beta}^s - \mathbf{I}^s \times \omega \cdot \{\mathbf{a}\}^T \dot{\beta}^s \end{aligned} \quad (113)$$

This result, when substituted into Eq. (105), would be appropriate for the simulation of the vehicle of Fig. 9 in the unlikely mode of motion in which the appendages are undergoing gradual change of orientation with respect to  $B_1$ ,

while the inertial angular velocity  $\omega$  of  $B_1$  is large. More useful forms of this result appear either when  $\Omega^a$  is identically zero, which leaves

$$\begin{aligned} \frac{d}{dt} \int_A \mathbf{p} \times \dot{\mathbf{p}} dm = & \omega \times [\mathbf{R} \times \{\mathbf{a}\}^T \Sigma m_s \dot{u}^s] + \mathbf{R} \times [\{\mathbf{a}\}^T \Sigma m_s \ddot{u}^s + \omega \times \{\mathbf{a}\}^T \Sigma m_s \dot{u}^s] \\ & + \omega \times (\Sigma \mathbf{r}^s \times \{\mathbf{a}\}^T m_s \dot{u}^s) + \Sigma \mathbf{r}^s \times [\{\mathbf{a}\}^T m_s \ddot{u}^s + \omega \times \{\mathbf{a}\}^T m_s \dot{u}^s] \\ & + \Sigma \mathbf{I}^s \cdot [\{\mathbf{a}\}^T \ddot{\beta}^s + \omega \times \{\mathbf{a}\}^T \dot{\beta}^s] + \omega \times \mathbf{I}^s \cdot \{\mathbf{a}\}^T \dot{\beta}^s - \mathbf{I}^s \times \omega \cdot \{\mathbf{a}\}^T \dot{\beta}^s \end{aligned} \quad (114)$$

or when  $\omega$  is also assumed to be small and included in the linearization, with the result

$$\frac{d}{dt} \int_A \mathbf{p} \times \dot{\mathbf{p}} dm = \mathbf{R} \times \{\mathbf{a}\}^T \Sigma m_s \ddot{u}^s + \Sigma \mathbf{r}^s \times \{\mathbf{a}\}^T m_s \ddot{u}^s + \mathbf{R} \times (\dot{\Omega}^a \times \Sigma m_s \mathbf{r}^s) + \Sigma m_s \mathbf{r}^s \times (\dot{\Omega}^a \times \mathbf{r}^s) + \Sigma \mathbf{I}^s \cdot \{\mathbf{a}\}^T \ddot{\beta}^s \quad (115)$$

These two special cases may be of value in practical applications, since they accommodate the tri-spin vehicle with flexible appendages on the de-spun body, and also the dual-spin vehicle with appendages on the spinning body. Even further specialization may be warranted, since spinning vehicles with flexible appendages are usually simple "spinners," rather than dual-spin vehicles, and a de-spun body with flexible appendages is more apt to be part of a dual-spin vehicle than of a tri-spin vehicle, in which case the relative motion  $\Omega^a$  is usually zero.

The essential differences among the expanded forms of the rotational equation (105) are simply differences in the size of the equations and the amount of bookkeeping involved in writing them in matrix form. Because the purpose here is simply to illustrate the structure of the vehicle equations, attention is restricted henceforth to a dual-spin vehicle with a rigid, symmetric rotor, a nutation damper, and a single flexible appendage without rotational capability relative to its base ( $\Omega^a \equiv 0$  and  $C \equiv E$  so  $\{\mathbf{a}\} = \{\mathbf{b}\}$ ). The platform angular velocity  $\omega$  is unrestricted, so the equations to follow may easily be specialized either to the simple spinner with flexible appendages (by omitting the rotor) or to the dual-spin vehicle with an appendage on a de-spun body (by linearizing in  $\omega$ ). By taking both of these steps (omitting the rotor and linearizing in  $\omega$ ), one obtains equations of motion of a space vehicle with three-axis active reaction-jet control in an inertially stationary nominal orientation.

By inspection of Eqs. (105), (107), (108), and (114), the required vector-dyadic equation is obtained as follows:

$$\begin{aligned} \mathbf{T} = & \mathbf{I} \cdot \dot{\omega} + \omega \times \mathbf{I} \cdot \omega + \dot{\mathbf{I}} \cdot \omega + \dot{\mathbf{h}} + m\mathbf{b}(\ddot{\xi}\mathbf{b}_2 + \dot{\xi}\omega \times \mathbf{b}_2) + \omega \times [\mathbf{R} \times \{\mathbf{b}\}^T \Sigma m_s \dot{u}^s] \\ & + \mathbf{R} \times [\{\mathbf{b}\}^T \Sigma m_s \ddot{u}^s + \omega \times \{\mathbf{b}\}^T \Sigma m_s \dot{u}^s] + \omega \times [\Sigma \mathbf{r}^s \times \{\mathbf{b}\}^T m_s \dot{u}^s] + \Sigma \mathbf{r}^s \times [\{\mathbf{b}\}^T m_s \ddot{u}^s + \omega \times \{\mathbf{b}\}^T m_s \dot{u}^s] \\ & + \Sigma \mathbf{I}^s \cdot [\{\mathbf{b}\}^T \ddot{\beta}^s + \omega \times \{\mathbf{b}\}^T \dot{\beta}^s] + \omega \times \Sigma \mathbf{I}^s \cdot \{\mathbf{b}\}^T \dot{\beta}^s - \Sigma \mathbf{I}^s \times \omega \cdot \{\mathbf{b}\}^T \dot{\beta}^s \end{aligned} \quad (116)$$

Note that the mass-center motion  $\mathbf{c}$  has been assumed to be infinitesimal, in accordance with the indicated specialization.

Terms involving  $\mathbf{I}$  in Eq. (116) are time-variable, since the inertia dyadic of the composite vehicle depends on its deformations. By definition (see Eq. 103), the vehicle inertia dyadic for point  $O$  is given by

$$\mathbf{I} = \int (\mathbf{p} \cdot \mathbf{p} \mathbf{E} - \mathbf{p}\mathbf{p}) dm \quad (117)$$

Let the generic symbol  $\mathbf{p}^*$  be the value of  $\mathbf{p}$  when the vehicle is undeformed, so that the inertia dyadic of the undeformed vehicle is

$$\mathbf{I}^* = \int (\mathbf{p}^* \cdot \mathbf{p}^* \mathbf{E} - \mathbf{p}^* \mathbf{p}^*) dm \quad (118)$$

For all points of the vehicle except those in the flexible appendage and the nutation damper mass,  $\mathbf{p}$  is identically  $\mathbf{p}^*$ . For points in the translating damper body  $B_2$ ,

$$\mathbf{p} = \mathbf{p}^* + \xi \mathbf{b}_1 = b\mathbf{b}_3 + \xi \mathbf{b}_1 \quad (119)$$

and for the mass center  $P_s$  of body  $A_s$  of the appendage,

$$\mathbf{p} = \mathbf{p}^* + \mathbf{u}^s = \mathbf{R} + \mathbf{r}^s + \mathbf{u}^s \quad (120)$$

In addition to the contribution of the translational deformation  $\mathbf{u}^s$  to the deformed vehicle inertia dyadic  $\mathbf{I}$ , there is a contribution from the rotational deformation  $\beta^s$  for each sub-body  $A_s$ . If the vector array  $\{\mathbf{a}^s\}$  consists of a dextral orthonormal set fixed in  $A_s$ , then the inertia dyadic  $\mathbf{I}^s$  of  $A_s$  referenced to the mass center  $P_s$  may be written in the  $\{\mathbf{a}^s\}$  basis as

$$\mathbf{I}^s = \{\mathbf{a}^s\}^T I^s \{\mathbf{a}^s\} \quad (121)$$

If now  $\{\mathbf{a}^s\}$  is aligned with  $\{\mathbf{b}\}$  when the appendage is undeformed, then the deformation  $\{\mathbf{b}\}^T \beta^s$  yields the unit

vector relationship

$$\{\mathbf{a}^s\} = (E - \tilde{\beta}^s) \{\mathbf{b}\} \quad (122)$$

since  $E - \tilde{\beta}^s$  is the linear approximation of the direction cosine matrix generated by the orthogonal 1-2-3-type rotations  $\beta_1, \beta_2, \beta_3$ . Thus in terms of the vector basis  $\{\mathbf{b}\}$ , the inertia dyadic for body  $A_s$  becomes

$$\begin{aligned} \mathbf{I}^s &= \{\mathbf{b}\}^T (E - \tilde{\beta}^s)^T I^s (E - \tilde{\beta}^s) \{\mathbf{b}\} \\ &= \{\mathbf{b}\}^T (E + \tilde{\beta}^s) I^s (E - \tilde{\beta}^s) \{\mathbf{b}\} \\ &= \{\mathbf{b}\}^T I^s \{\mathbf{b}\} + \{\mathbf{b}\}^T (\tilde{\beta}^s I^s - I^s \tilde{\beta}^s) \{\mathbf{b}\} \end{aligned} \quad (123)$$

The second term is therefore the contribution of rotation  $\beta^s$  to the inertia dyadic  $\mathbf{I}$  of the vehicle.

From Eqs. (117) through (120) and (123), it follows that the vehicle inertia dyadic is

$$\mathbf{I} = \mathbf{I}^* - mb\xi(\mathbf{b}_3\mathbf{b}_1 + \mathbf{b}_1\mathbf{b}_3) + \Sigma m_s [2(\mathbf{R} + \mathbf{r}^s) \cdot \mathbf{u}^s \mathbf{E} - (\mathbf{R} + \mathbf{r}^s) \mathbf{u}^s - \mathbf{u}^s (\mathbf{R} + \mathbf{r}^s)] + \Sigma \{\mathbf{b}\}^T (\tilde{\beta}^s I^s - I^s \tilde{\beta}^s) \{\mathbf{b}\} \quad (124)$$

Noting that the identity dyadic is

$$\mathbf{E} = \{\mathbf{b}\}^T \{\mathbf{b}\} \quad (125)$$

so that, for example,

$$\mathbf{R} \cdot \mathbf{u}^s \mathbf{E} = R^T \{\mathbf{b}\} \cdot \{\mathbf{b}\}^T u^s \{\mathbf{b}\}^T \{\mathbf{b}\} = R^T u^s \{\mathbf{b}\}^T \{\mathbf{b}\} = \{\mathbf{b}\}^T R^T u^s \{\mathbf{b}\}$$

one may write the inertia dyadic as

$$\begin{aligned} \mathbf{I} &= \mathbf{I}^* - \{\mathbf{b}\}^T mb\xi(E^1 E^{3T} + E^3 E^{1T}) \{\mathbf{b}\} \\ &\quad + \{\mathbf{b}\}^T \Sigma m_s [2(R + r^s)^T u^s E - (R + r^s) u^{sT} - u^s (R + r^s)^T] \{\mathbf{b}\} + \{\mathbf{b}\}^T \Sigma (\tilde{\beta}^s I^s - I^s \tilde{\beta}^s) \{\mathbf{b}\} \end{aligned} \quad (126)$$

The derivative  $\dot{\mathbf{I}}$  is readily obtained from Eq. (126) as

$$\dot{\mathbf{I}} = \{\mathbf{b}\}^T \{-mb\xi(E^1 E^{3T} + E^3 E^{1T}) + \Sigma m_s [2(R + r^s)^T \dot{u}^s E - (R + r^s) \dot{u}^{sT} - \dot{u}^s (R + r^s)^T] + \Sigma (\tilde{\beta}^s I^s - I^s \tilde{\beta}^s)\} \{\mathbf{b}\} \quad (127)$$

Finally, the complete and explicit vector-dyadic equation of rotational motion can be obtained by substituting Eqs. (126) and (127) into Eq. (116). In the process, all vectors and dyadics are written in terms of the vector array  $\{\mathbf{b}\}$ , and the matrix representation of cross-products is employed (see Eq. 44). The result is

$$\begin{aligned} \{\mathbf{b}\}^T T &= \{\mathbf{b}\}^T \{I^* \dot{\omega} - mb\xi(E^1 E^{3T} + E^3 E^{1T}) \dot{\omega} + \Sigma m_s [2(R + r^s)^T u^s E - (R + r^s) u^{sT} - u^s (R + r^s)^T] \dot{\omega} + \Sigma (\tilde{\beta}^s I^s - I^s \tilde{\beta}^s) \dot{\omega} \\ &\quad + \tilde{\omega} I^* \omega - \tilde{\omega} mb\xi(E^1 E^{3T} + E^3 E^{1T}) \omega + \tilde{\omega} \Sigma m_s [2(R + r^s)^T u^s E - (R + r^s) u^{sT} - u^s (R + r^s)^T] \omega \\ &\quad + \tilde{\omega} \Sigma (\tilde{\beta}^s I^s - I^s \tilde{\beta}^s) \omega - mb\xi(E^1 E^{3T} + E^3 E^{1T}) \omega + \Sigma m_s [2(R + r^s)^T \dot{u}^s E - (R + r^s) \dot{u}^{sT} - \dot{u}^s (R + r^s)^T] \omega \\ &\quad + \Sigma (\tilde{\beta}^s I^s - I^s \tilde{\beta}^s) \omega + \dot{h} + \tilde{\omega} h + mb\xi \ddot{E}^2 + mb\xi \tilde{\omega} E^2 + \tilde{\omega} \tilde{R} \Sigma m_s \dot{u}^s + \tilde{R} \Sigma m_s \dot{u}^s + \tilde{R} \tilde{\omega} \Sigma m_s \dot{u}^s + \tilde{\omega} \Sigma r^s m_s \dot{u}^s \\ &\quad + \Sigma \tilde{r}^s (m_s \ddot{u}^s + \tilde{\omega} m_s \dot{u}^s) + \Sigma I^s (\ddot{\beta}^s + \tilde{\omega} \dot{\beta}^s) + \tilde{\omega} \Sigma I^s \dot{\beta}^s - \Sigma I^s \tilde{\omega} \dot{\beta}^s\} \end{aligned} \quad (128)$$

Symbols  $T$ ,  $h$ , and  $I^*$  are new, being defined here by their context. The identities

$$\mathbf{I}^s \times \boldsymbol{\omega} \cdot \{\mathbf{b}\}^T \dot{\beta}^s = \mathbf{I}^s \cdot \boldsymbol{\omega} \times \{\mathbf{b}\}^T \dot{\beta}^s = \{\mathbf{b}\}^T I^s \dot{\omega} \dot{\beta}^s$$

and

$$\dot{\mathbf{h}} = \ddot{\mathbf{h}} + \boldsymbol{\omega} \times \mathbf{h} = \{\mathbf{b}\}^T \dot{h} + \{\mathbf{b}\}^T \omega h$$

have been employed in writing Eq. (128).

The matrix equation isomorphic to the vector equation (128) can be written, by inspection, simply by dropping the vector arrays (or by pre-dot-multiplying by  $\{\mathbf{b}\}^T$ ). This matrix equation is more useful when written in terms of the total  $6n$  by  $1$  deformation matrix  $q$  (defined in Eq. 83), making use of the operators  $\Sigma_{E0}$  and  $\Sigma_{0E}$  (see Eq. 85). The result then adopts the form

$$\begin{aligned} T = & I^* \dot{\omega} + \tilde{\omega} I^* \omega + \dot{h} + \tilde{\omega} h + 2 \Sigma_{E0}^T M (\Sigma_{E0} R + r)^T q \Sigma_{E0} \dot{\omega} - \Sigma_{E0}^T M (\Sigma_{E0} R + r) q^T \Sigma_{E0} \dot{\omega} \\ & - \Sigma_{E0}^T M q (\Sigma_{E0} R + r)^T \Sigma_{E0} \dot{\omega} + \Sigma_{E0}^T (\tilde{q} M - M \tilde{q}) \Sigma_{0E} \dot{\omega} - m b \xi (E^1 E^{3T} + E^3 E^{1T}) \dot{\omega} + m b (\ddot{\xi} E + \dot{\xi} \tilde{\omega}) E^2 \\ & - m b (\dot{\xi} E + \xi \tilde{\omega}) (E^1 E^{3T} + E^3 E^{1T}) \omega + \tilde{\omega} \tilde{R} \Sigma_{E0}^T M \dot{q} + \tilde{R} \Sigma_{E0}^T M \ddot{q} + \tilde{R} \tilde{\omega} \Sigma_{E0}^T M \dot{q} + \tilde{\omega} \Sigma_{E0}^T \tilde{r} M \dot{q} + \Sigma_{E0}^T \tilde{r} M \ddot{q} \\ & + \Sigma_{0E}^T M \ddot{q} + \tilde{\omega} \Sigma_{0E}^T M \dot{q} + \Sigma_{E0}^T \tilde{r} \Sigma_{E0} \tilde{\omega} \Sigma_{E0}^T M \dot{q} + \Sigma_{E0}^T M \Sigma_{0E} \tilde{\omega} \Sigma_{0E}^T \dot{q} - \tilde{\omega} \Sigma_{0E}^T M \Sigma_{0E} \tilde{\omega} \Sigma_{0E}^T \dot{q} + \Sigma_{0E}^T \tilde{q} M \Sigma_{0E} \omega \\ & - \Sigma_{0E}^T M \tilde{q} \Sigma_{0E} \omega + \tilde{\omega} \Sigma_{0E}^T \tilde{q} M \Sigma_{0E} \omega - \tilde{\omega} \Sigma_{E0}^T \tilde{q} M \Sigma_{0E} \omega + 2 \tilde{\omega} \Sigma_{E0}^T M [(\Sigma_{E0} R + r)^T q] \Sigma_{E0} \omega \\ & - \tilde{\omega} \Sigma_{E0}^T M (\Sigma_{E0} R + r) q^T \Sigma_{E0} \omega - \tilde{\omega} \Sigma_{E0}^T M q (\Sigma_{E0} R + r)^T \Sigma_{E0} \omega + 2 \Sigma_{E0}^T M (\Sigma_{E0} R + r)^T \dot{q} \Sigma_{E0} \omega \\ & - \Sigma_{E0}^T [(\Sigma_{E0} R + r) \dot{q}^T - \dot{q}^T (\Sigma_{E0} \tilde{R} + r)^T] \Sigma_{E0} \omega \end{aligned} \quad (129)$$

The derivation of this equation has been the principal objective in this section. Interpretation may be facilitated by restricting attention to the special case of a dual-spin vehicle with an appendage on a nominally de-spun platform. This case is significant from an applications viewpoint, and yet the assumption that  $\omega$  is infinitesimal renders Eq. (129) quite interpretable. Since  $\omega$  is small, it can be replaced by the matrix  $\dot{\theta}$ , where  $\dot{\theta} = [\dot{\theta}_1 \ \dot{\theta}_2 \ \dot{\theta}_3]^T$ . The result is simply the linear equation

$$\begin{aligned} T = & I^* \ddot{\theta} + \dot{h} - \tilde{h} \dot{\theta} + m b \xi \ddot{E}^2 + \tilde{R} \Sigma_{E0}^T M \ddot{q} \\ & + \Sigma_{E0}^T \tilde{r} M \dot{q} + \Sigma_{0E}^T M \ddot{q} \end{aligned} \quad (130)$$

Since  $I^*$  is the inertia matrix of the total vehicle as a rigid body, the equation  $T = I^* \ddot{\theta}$  would suffice if the rotor were fixed (nonrotating) and the appendage were rigid. The torque applied to the body  $B_1$  in accelerating the relative angular rate of the rotor is  $-\dot{h}$ , and the "gyroscopic stiffness" supplied by the rotor is in  $-\tilde{h} \dot{\theta}$ . The term  $m b \xi \ddot{E}^2$  reflects the "inertial torque" due to damper mass acceleration, and the last three terms reflect the effects of appendage deformation.

The more general Eq. (129) has the major added term  $\tilde{\omega} I^* \omega$ , which introduces the "Euler coupling" or "gyroscopic coupling" of the vehicle as a spinning rigid body. In addition, Eq. (129) includes a multitude of nonlinear terms that reflect the influence of the vehicle deformability. They can be categorized as various kinds of acceleration terms (centripetal, coriolis, etc.), but perhaps they are best understood by reviewing their origins in the vector-dyadic equation (116).

The matrix equation of general rotation of the vehicle, Eq. (129), is compatible in its assumptions with Eq. (95), which defines the appendage response to a given arbitrary base motion. These equations do not yet constitute a complete set, however, since they include (in addition to unspecified external forces and torques) the unknown motion variables  $\ddot{X}$ ,  $\xi$ , and  $h$ .

The  $3$  by  $1$  matrix  $\ddot{X}$  is the matrix in an inertial vector basis of the inertial acceleration vector of the vehicle mass center. This vector is easily related to the resultant applied force  $\mathbf{F}$  by applying Newton's second law, i.e.,

$$\mathbf{F} = m \ddot{\mathbf{X}} = m \{\mathbf{i}\}^T \ddot{\mathbf{X}}$$

If  $\mathbf{F}$  is written in terms of the body-fixed vector basis  $\{\mathbf{b}\}$ , related to  $\{\mathbf{i}\}$  as in Eq. (53), this equation becomes

$$\{\mathbf{b}\}^T \mathbf{F} = \eta \{\mathbf{i}\}^T \ddot{\mathbf{X}} = \eta \{\mathbf{b}\}^T \Theta \ddot{\mathbf{X}}$$

or in matrix terms

$$\mathbf{F} = \eta \Theta \ddot{\mathbf{X}} \quad (131)$$

Newton's second law also provides the equation of motion of the damped linear oscillation representing the

rotation damper, except that only the  $\mathbf{b}_1$  component of the vector equation of translation is required. The vector result is

$$M(\ddot{\mathbf{X}} + \ddot{\mathbf{c}} + b\ddot{\mathbf{b}}_3 + \ddot{\xi}\mathbf{b}_1 + \xi\ddot{\mathbf{b}}_1) \cdot \mathbf{b}_1 = \mathbf{F}^{B_1} \cdot \mathbf{b}_1 = -k\xi - d\dot{\xi} \quad (132)$$

where  $k$  and  $d$  are scalar constants of the spring and dashpot, respectively. The mass-center displacement vector  $\mathbf{c}$  is available from Eq. (60) as

$$\mathbf{c} = -\{\mathbf{b}\}^T \Sigma_{E0} M \mathbf{q} - (m\xi/\eta) \mathbf{b}_1 = -\{\mathbf{b}\}^T [\Sigma_{E0} M \mathbf{q} + m\xi E^1/\eta]$$

so that

$$\mathbf{c} = -\{\mathbf{b}\}^T [\Sigma_{E0} \mu \ddot{\mathbf{q}} + m\ddot{\xi} E^1/\eta + 2\omega(\Sigma_{E0} \mu \dot{\mathbf{q}} + m\dot{\xi} E^1/\eta) + \ddot{\omega}(\Sigma_{E0} \mu \mathbf{q} + m\xi E^1/\eta) + \omega\ddot{\omega}(\Sigma_{E0} \mu \mathbf{q} + m\xi E^1/\eta)] \quad (133)$$

Similar operations on  $\mathbf{b}_3 = \{\mathbf{b}\}^T E^3$  and  $\mathbf{b}_1 = \{\mathbf{b}\}^T E^1$  permit the expansion of the term in parentheses in Eq. (132), and the  $\mathbf{b}_1$  dot-product then provides the scalar equation

$$m(1 - m/\eta) \ddot{\xi} + (d - 2mE^{1T} \omega E^1 m/\eta) \dot{\xi} + [k - mE^{1T} (\ddot{\omega} + \omega\ddot{\omega})(1 - m/\eta) E^1] \xi = mE^{1T} [-\Theta \ddot{\mathbf{X}} - b(\ddot{\omega} + \omega\ddot{\omega}) E^3 + \Sigma_{E0} \mu \ddot{\mathbf{q}} + 2\omega \Sigma_{E0} \mu \dot{\mathbf{q}} + (\ddot{\omega} + \omega\ddot{\omega}) \Sigma_{E0} \mu \mathbf{q}] \quad (134)$$

It should be noted that the ratio  $m/\eta$  is too small to warrant retention when  $B_2$  is a nutation damper, although for other applications of translating second bodies this may not be the case.

Finally, an equation of motion must be provided to determine the remaining variable  $h$ , the angular momentum matrix of the rotor  $B_3$  relative to body  $B_1$ , expressed in the vector basis of  $B_1$ . Because the rotor spins about its axis of symmetry, its relative angular momentum can always be expressed as the product of a scalar  $\mathcal{I}\dot{\phi}$  and a unit vector fixed in  $B_1$ , where  $\mathcal{I}$  is the moment of inertia of the rotor about its symmetry axis and  $\dot{\phi}$  is the spin rate of the rotor  $B_3$  relative to the base  $B_1$ . The orientation of the rotor in  $B_1$  is here selected so that the rotor axis parallels  $\mathbf{b}_3$ , merely for analytical convenience. Thus the relative angular momentum  $h$  may be expressed as

$$h = \mathcal{I}\dot{\phi} E^3 \quad (135)$$

Thus a single scalar equation must be derived for the determination of the scalar unknown  $\dot{\phi}$ . This is readily obtained as the scalar Euler equation corresponding to the axis of symmetry, namely,

$$\tau = \mathcal{I}(\ddot{\phi} + \dot{\omega}_3) = \mathcal{I}(\ddot{\phi} + E^{3T} \dot{\omega}) \quad (136)$$

where  $\tau$  is the magnitude of the torque applied to the rotor along its axis of symmetry. In general  $\tau$  is some combination of bearing friction and the motor torque established by a control law. Complete simulation would of course require an auxiliary equation for that control law, and except for very preliminary calculations the control law would be nonlinear.

In summary, Eqs. (136), (134), (131), (129), and (95) constitute a complete set of dynamic equations for the system. The unknowns are included in  $\phi$ ,  $\xi$ ,  $\mathbf{X}$ ,  $\Theta$ ,  $\omega$ , and  $\mathbf{q}$ . (Equation 135 must be substituted into Eq. 129 before collecting these equations, in order to eliminate  $h$ .) To complete the simulation, a set of kinematic equations relating  $\Theta$  and  $\omega$  must be included. The particular equations used will depend on the attitude parameters chosen to define the orientation of  $B_1$  in inertial space (e.g., direction cosines, Euler parameters, attitude angles, etc.). These equations are available from the fundamental kinematic relationship (Ref. 16)

$$\ddot{\omega} = \dot{\Theta} \Theta^T = -\Theta \dot{\Theta}^T \quad (137)$$

but more suitable forms of the kinematic equations can be obtained for specific parameter choices (e.g., see Ref. 23).

It is then conceptually possible to collect the five dynamic equations (136), (134), (131), (129), and (95) together with kinematic equations from Eq. (137) and control laws for the rotor torque  $\tau$  and the vehicle torque  $T$ , and with this set, to accomplish the simulation of a dual-spin vehicle with a flexible appendage and a nutation damper attached to a base undergoing arbitrary motion. By specializing these equations, one could simulate a simple spin-stabilized vehicle, or a vehicle with three-axis reaction-jet control. By generalizing these equations, proceeding from vector-dyadic equations developed here in detail, one could with straightforward labor similarly simulate a vehicle controlled by several rotors ("momentum wheels") and having articulated elastic appendages capable of substantial relative motion. All of this is, however, only *conceptually* possible. The complete set of equations is available, but if they are to be applied to a real space vehicle represented by a reasonably accurate mathematical model, the dimension of the resulting equations is prohibitively high, even for machine computation. Remember that Eq. (95) alone consists, for practical problems, of hundreds of second-order scalar differential equations, and that the entire system of equations is nonlinear and highly coupled. If these equations are to have practical value, they must be subjected to coordinate transformations that accomplish substantial uncoupling of the equations and permit the analyst to exercise judgment in restricting his attention to a subset of the present set of perhaps hundreds of coordinates, in order to justify working with a more reasonable number of differential equations in the simulation of the vehicle.

#### D. Coordinate Transformations

**1. Transformation rationale.** The objective of this section is to investigate the possibility of simplifying the equations of motion by introducing linear transformations for some of the variables. The transformations are applied to the appendage deformation matrix  $q$  only, leaving the discrete coordinates of the base  $B_1$ , the damper  $B_2$ , and the rotor  $B_3$  unchanged. The more remote possibility of finding a transformation that may fruitfully be applied to the entire system of coordinates is treated in Section IV.

Of course, an infinite variety of coordinate transformations may be devised, even within the linear class (to which attention is presently restricted). Any transformation that provides a one-to-one mapping from one coordinate system to the next yields equations that could theoretically provide a correct simulation, but obviously not all such transformations are advantageous. It may be, however, that more than one useful transformation will

be available, and then the appropriate choice may depend on individual standards of utility.

The basic difficulty in using the equations of motion derived earlier (Eqs. 95, 129, 131, 134, and 136 or their equivalent) is simply their dimension. A certain amount of nonlinearity and coupling of equations may be unavoidable. Therefore the primary test of the utility of a given coordinate transformation is the degree to which it permits the truncation of the coordinate matrix, and the consequent reduction in the dimension of the system of equations.

Coordinate truncation is never a completely rigorous process, since it results in an incomplete and imprecise indication of the response of the mathematical model to its dynamic environment. As a practical matter, however, it must be recognized that the mathematical model is but an imperfect approximation of the vehicle being studied, so it is sophistry to argue that coordinate truncation necessarily degrades the simulation of a real vehicle. Some degree of truncation may be appropriate even when the most realistic efficient simulation is sought.

There are no entirely satisfactory analytical procedures for determining the degree to which a given matrix of coordinates can safely be truncated, nor can the relative acceptability of truncation of alternative coordinate systems be unequivocally established. It should be noted, however, that when a coordinate transformation provides a completely uncoupled system of scalar differential equations, one can obtain the solution of these equations precisely for each scalar coordinate in turn, without considering the influence of other coordinates. Therefore one can truncate the coordinate matrix without sacrificing the rigorous validity of the solutions for the coordinates retained. The determination of the *number* of such coordinates to retain for acceptable simulation is still a matter of engineering judgment, but the validity of the solution for those retained is related directly to the degree to which the transformed equations are uncoupled. Accordingly, the desirability of a given coordinate transformation is measured here in terms of the degree to which it uncouples the system of differential equations employed in the vehicle simulation.

The total system of equations required for the vehicle simulation includes some equations in which the appendage deformation terms are of central importance (e.g., the appendage equation 95), some equations in which terms in  $q$  are generally of secondary importance (e.g., the nutation damper equation 134), in which  $q$  is always multi-

plied by the small constant matrix  $\mu$ ), and some equations in which  $q$  does not appear at all (e.g., the rotor equation 136). In the quest for a transformation that to some degree uncouples the equations, it is therefore reasonable to concentrate on a subset of the system of differential equations, excluding from consideration the equations of motion of rotor and damper, the kinematic equations relating  $\omega$  and  $\Theta$ , and the control system equations. (Note that the last of these will include  $q$  when sensors or thrusters are mounted on flexible appendages.) This decision to concentrate on certain equations clearly marks a compromise with the objective of securing uncoupled equations, and, in fact, this objective became unrealizable with the announced restriction in this section to transformations of the appendage deformation coordinates  $q$  only. The final equations must certainly remain coupled by the

discrete coordinates of  $B_1$ ,  $B_2$ , and  $B_3$ , and by ignoring certain equations while transformations are sought, one accepts as well the likelihood that these equations will, after transformation, remain coupled in the new deformation variables.

Attention is now restricted to Eqs. (95), (129), and (131), which are respectively the appendage equation, the vehicle rotation equation, and the vehicle translation equation. The last of these is trivial, permitting the solution

$$\Theta \ddot{X} = F/\mathcal{M} \quad (138)$$

This substitution into Eq. (95) with the substitution  $\ddot{e} = -m\ddot{\xi}E^1/\mathcal{M}$  to accommodate the damper, provides the revised appendage equation

$$\begin{aligned} M(E - \Sigma_{E0} \Sigma_{E0}^T M/\mathcal{M}) \ddot{q} + 2\{M[(\Sigma_{E0} \omega)^\sim - \Sigma_{E0} \tilde{\omega} \Sigma_{E0}^T M/\mathcal{M}] + M(\Sigma_{0E} \omega)^\sim + (\Sigma_{0E} \omega)^\sim M - (M \Sigma_{0E} \omega)^\sim + D'\} \dot{q} \\ - \{M[(\Sigma_{E0} \dot{\omega})^\sim - \Sigma_{E0} \dot{\omega} \Sigma_{E0}^T M/\mathcal{M}] + M[(\Sigma_{E0} \omega)^\sim (\Sigma_{E0} \omega)^\sim - \Sigma_{E0} \tilde{\omega} \tilde{\omega} \Sigma_{E0}^T M/\mathcal{M}] + K\} q = \\ - M(\Sigma_{0E} - \Sigma_{E0} \tilde{R} - \tilde{r} \Sigma_{E0}) \dot{\omega} - M[\Sigma_{E0} \tilde{\omega} \tilde{\omega} R - (\Sigma_{E0} \omega)^\sim (\Sigma_{E0} \omega)^\sim \tilde{r}] - (\Sigma_{0E} \omega)^\sim M(\Sigma_{0E} \omega) \\ - M \Sigma_{E0} F/\mathcal{M} + M \Sigma_{E0} m\ddot{\xi}E^1/\mathcal{M} + \lambda \end{aligned} \quad (139)$$

This equation consists of  $6n$  second-order scalar equations and can be written as a matrix equation of the structure of Eq. (84), or somewhat more explicitly as

$$M' \ddot{q} + D' \dot{q} + G' q + K' q + A' q = L' = -M(\Sigma_{0E} - \Sigma_{E0} \tilde{R} - \tilde{r} \Sigma_{E0}) \dot{\omega} + N' - M \Sigma_{E0} F/\mathcal{M} + M \Sigma_{E0} m\ddot{\xi}E^1/\mathcal{M} + \lambda \quad (140)$$

where  $N'$  represents the nonlinear terms in  $\omega$  due to centrifugal forces on the appendage. Here  $G'$  depends linearly on  $\omega$ ,  $A'$  depends linearly on  $\dot{\omega}$ , and  $K'$  depends nonlinearly on  $\omega$  (involving only second-degree terms). As noted in the discussion of Eq. (84),  $M'$ ,  $D'$ , and  $K'$  are symmetric and  $G'$  and  $A'$  are skew-symmetric.

The remaining equation of primary interest, Eq. (129), consists of only three scalar equations, although many of its terms involve matrix operators that accomplish summations over  $6n$  elements. The appearance of the variable  $q$  and its derivatives in Eq. (129) is again linear, and again  $\dot{\omega}$  appears only in the first power and  $\omega$  in the second, so it would be possible to write this equation in the form of Eq. (84). This would be misleading, however, since the coefficient matrices would be rectangular (3 by  $6n$ ), and the role of Eq. (129) as the vehicle rotational equation would be obscured. To emphasize the primary physical significance of this equation, it is rewritten in symbolic form as

$$\begin{aligned} I \dot{\omega} + \tilde{\omega} I \omega + (\dot{I} + G - \tilde{h}) \omega = \\ - (\Sigma_{0E}^T + \tilde{R} \Sigma_{0E}^T + \Sigma_{E0}^T \tilde{r}) M \ddot{q} - \dot{h} - m b \ddot{\xi} E^2 + T \end{aligned} \quad (141)$$

This form of the vehicle rotational equation can be constructed from Eq. (129) quite mechanically by defining  $I$  as the coefficient matrix of  $\dot{\omega}$ , observing that this same matrix permits  $\tilde{\omega} I \omega$  to accommodate all second-degree terms in  $\omega$ , and then defining  $G$  to include all terms in the coefficient matrix of  $\omega$  not absorbed by  $\dot{I}$  or  $\tilde{h}$ . Much more insight into the significance of  $I$  and  $G$  may be gained by inspecting Eq. (116), which is the vector-dyadic antecedent of Eq. (129). With this examination comes the realization that  $I$  is the time-variable inertia matrix of the total vehicle in basis  $\{b\}$ , referred to the vehicle mass center. The term  $G\omega$  accommodates the "inertial torques" applied to the vehicle due to the coriolis accelerations of appendage and damper, so it has the same physical origins as the matrix  $G'$  of Eq. (140). It should be noted that



the coefficient of  $\ddot{q}$  in Eq. (141) is the transpose of the coefficient of  $\dot{\omega}$  in Eq. (140), i.e.,

$$[(\Sigma_{0R}^T + \tilde{R} \Sigma_{R0}^T + \Sigma_{R0}^T \tilde{R}) M]^T = M^T (\Sigma_{0R} + \Sigma_{R0} \tilde{R}^T + \tilde{R}^T \Sigma_{R0}) \\ = M (\Sigma_{0R} - \Sigma_{R0} \tilde{R} - \tilde{R} \Sigma_{R0}) \quad (142)$$

since a symmetric matrix is its own transpose and a skew-symmetric matrix is the negative of its transpose.

Equations (140) and (141) are coupled with each other, and Eq. (141) is also coupled with the equations of motion of rotor and damper. Nonetheless, physically based arguments can be introduced that tend to justify the use of a transformation on  $q$ , which uncouples the new deformation variables in Eq. (140) only, leaving linear combinations of these variables in each of the scalar equations of Eq. (141). Truncation of the new deformation coordinate matrix would then permit substantial reduction of the number of scalar equations accepted from Eq. (140), and this would provide exact solutions for the coordinates retained only insofar as  $\omega$  (as obtained from Eq. 141) is uninfluenced by the excluded coordinates. This seems quite an acceptable approximation, since in general the appendage vibrations act only as perturbations on the motion of the controlled vehicle, which is dominated by the control torque  $T$  in Eq. (141).

Finally, then, attention focuses on the quest for a coordinate transformation that uncouples the deformation coordinates in the  $6n$  equations of Eq. (140). Only the homogeneous equation

$$M' \ddot{q} + D' \dot{q} + G' q + K' q + A' q = 0 \quad (143)$$

is relevant to this search.

Although matrices  $M'$  and  $D'$  are constant, matrices  $G'$ ,  $A'$ , and  $K'$  may vary with time, since they depend on  $\omega$  and  $\dot{\omega}$ . Of course  $\omega$  and  $\dot{\omega}$  are strictly unknown, but it is quite reasonable in the present context of space vehicle attitude-control-system simulation to assume that  $\omega$  varies only slightly from the nominal value (say,  $\omega(t)$ ) which is the objective of the control system. Under this assumption, formal linearization removes the unknown  $\omega - \omega$  from the homogeneous Eq. (143), since this small quantity appears in that equation only as a factor of  $\dot{q}$  or  $q$ . Thus one may argue quite generally that Eq. (143) is a system of linear second-order equations with coefficients depending *explicitly* on time.

No general procedures are available for the transformation of Eq. (143) into a system of uncoupled equations. This can be accomplished in general terms only when the coefficient matrices are constant, which requires that the nominal value  $\omega$  be constant. This is surely the most common situation of interest in spacecraft control, but it excludes certain appendage deployment and transient operations problems.

**2. State-equation modal analysis.** Consider then the special case of Eq. (143) for which  $M'$ ,  $D'$ ,  $G'$ , and  $K'$  are constant, implying  $A' = 0$ . This second-order matrix equation can always be represented by a first-order equation (the state equation). This is accomplished by defining the  $12n$  by  $1$  matrix

$$Q = \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \quad (144)$$

and writing

$$\dot{Q} = BQ \quad (145)$$

where

$$B = \begin{bmatrix} 0 & E \\ -(M')^{-1} K' & -(M')^{-1} (G' + D') \end{bmatrix} \quad (146)$$

Here  $B$  is a  $12n$  by  $12n$  matrix, and each of the partitions shown in Eq. (146) is of dimension  $6n$  by  $6n$ . Equation (145) has exactly the same content as Eq. (143), since it contains this equation in addition to the identity  $\dot{q} = \dot{q}$ .

Linear, constant-coefficient differential equations such as Eq. (145) always have solutions of the form

$$Q = \Phi^m e^{\lambda_m t} \quad (147)$$

where  $\lambda_m$  is a scalar and  $\Phi^m$  is a  $12n$  by  $1$  matrix, which from Eq. (144) must partition into a  $6n$  by  $1$  matrix  $\phi^m$  above a  $6n$  by  $1$  matrix  $\phi^m \lambda_m$ , i.e.,

$$\Phi^m = [\phi_1^m \ \phi_2^m \ \cdots \ \phi_{6n}^m \ (\phi_1^m \lambda_m) \ \cdots \ (\phi_{6n}^m \lambda_m)]^T = \begin{bmatrix} \phi^m \\ \phi^m \lambda_m \end{bmatrix} \quad (148)$$

The validity of the solution indicated in Eq. (147) can be confirmed by substitution into Eq. (145), yielding

$$\lambda_m \Phi^m = B \Phi^m$$

since  $e^{\lambda_m t} \neq 0$ . This result may be written in the conventional form of an eigenvalue problem,

$$(B - \lambda_m E) \Phi^m = 0 \quad (149)$$

so that  $\lambda_m$  is an eigenvalue and  $\Phi^m$  an eigenvector of  $B$ . The existence of a solution as in Eq. (147) follows from the existence of at least one eigenvalue of  $B$  and its corresponding eigenvector.

In scalar terms, Eq. (149) represents a set of  $12n$  homogeneous algebraic equations in the  $12n + 1$  unknowns  $\Phi_1^m, \dots, \Phi_{12n}^m$  and  $\lambda_m$ . Nontrivial (nonzero) solutions for  $\Phi^m$  exist only when  $\lambda_m$  takes on certain characteristic values or eigenvalues such that the determinant of coefficients is zero, by Cramer's rule. In other words, values of  $\lambda_m$  must be selected that satisfy

$$|B - \lambda_m E| = 0 \quad (150)$$

There are  $12n$  solutions  $\lambda_1, \dots, \lambda_{12n}$  for Eq. (150), as may be confirmed by expanding the determinant into the polynomial

$$b_{12n} \lambda^{12n} + b_{12n-1} \lambda^{12n-1} + \dots + b_2 \lambda^2 + b_1 \lambda + b_0 = 0$$

$$(\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_{12n}) = 0 \quad (151)$$

The matrix  $B$  and the related coefficients  $b_0, \dots, b_{12n}$  are real numbers for equations that stem from the dynamic equations of Eq. (143). Therefore the complex eigenvalues among  $\lambda_1, \dots, \lambda_{12n}$  occur in complex conjugate pairs. Subscripts may be assigned so that  $\lambda_{nn+m} = \lambda_m^*$  (asterisk here denotes complex conjugate).

Corresponding to each distinct eigenvalue  $\lambda_m$ , there exists an eigenvector  $\Phi^m$  that can be determined to within a multiplicative constant by solving Eq. (146). A completely unique solution is not possible, since the system of equations in Eq. (149) includes only  $12n - 1$  independent algebraic equations in the  $12n$  unknowns  $\Phi_1^m, \dots, \Phi_{12n}^m$ , with  $\lambda_m$  specified by Eq. (150). When the eigenvalues are distinct, the eigenvectors are independent (see Ref. 33, pp. 184 ff.), but when there are repeated eigenvalues, the corresponding eigenvectors may not be independent.

The validity of Eq. (149) for a particular  $\lambda_m$  and  $\Phi^m$  guarantees as well the validity of

$$(B - \lambda_m^* E) \Phi^{m*} = 0 \quad (152)$$

since the complex conjugate of zero is zero, and the conjugate of a product is the product of conjugates. Because  $\lambda_m^* = \lambda_{nn+m}$ , the eigenvectors corresponding to complex pairs of eigenvalues are, from Eq. (152), also complex conjugate pairs.

Although the solution to Eq. (145) has readily been found, and certain of the properties of the solution have been noted, these steps have not yet accomplished the main objective, which is the discovery of a transformation matrix that uncouples the coordinates in Eq. (143). It is in most cases not essential that the result be a second-order differential equation; it may suffice to obtain a first-order equation such as Eq. (145), but with a diagonal coefficient matrix. Although it is possible to generate a transformation from Eq. (143) to an uncoupled system of second-order equations when  $M'$  and  $K'$  are symmetric and  $G'$  is skew-symmetric, it is possible to diagonalize the state equation (145) even more generally. The latter transformation is considered first.

If  $\Phi$  is defined as the  $12n$  by  $12n$  matrix whose columns are the eigenvectors  $\Phi^1, \dots, \Phi^{12n}$  of the matrix  $B$ , then by inspection of Eq. (148) one may write

$$B\Phi = B[\Phi^1 \Phi^2 \dots \Phi^{12n}] = [B\Phi^1 \dots B\Phi^{12n}]$$

$$= [\lambda_1 \Phi^1 \dots \lambda_{12n} \Phi^{12n}]$$

$$= [\Phi^1 \dots \Phi^{12n}] \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_{12n} \end{bmatrix}$$

$$= \Phi \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{12n} \end{bmatrix} \quad (153)$$

It follows that, if  $\Phi^{-1}$  exists, premultiplication provides the diagonal matrix

$$\Phi^{-1} B \Phi = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{12n} \end{bmatrix} \quad (154)$$

The existence of  $\Phi^{-1}$  is assured if the determinant of  $\Phi$  is nonzero, i.e., if the eigenvectors  $\Phi^1, \dots, \Phi^{12n}$  are independent. It has been noted that this independence is guaranteed when  $\lambda_1, \dots, \lambda_{12n}$  are distinct. For the dynamic system of Eq. (143), the independence of the eigenvectors can be assured even without this restriction when  $M'$ ,  $G'$ , and  $K'$  are constant and  $A'$  and  $D'$  are zero. The necessary supporting argument is briefly summarized in the following paragraphs.

When the eigenvectors are independent, the general solution to Eq. (145) is the linear combination of solutions of the structure of Eq. (147), i.e.,

$$Q = \sum_{m=1}^{12n} A_m \phi^m e^{\lambda_m t} \quad (155)$$

The scalar constants  $A_m$  are established by initial conditions. It follows from Eq. (155) that if the real parts of  $\lambda_1, \dots, \lambda_{12n}$  are nonpositive, the response to initial conditions providing sufficiently small values of  $A_1, \dots, A_{12n}$  is a solution  $Q$  with an arbitrarily small norm. In other words, when the real parts of the eigenvalues of  $B$  are nonpositive, the null solution of Eq. (145) is Liapunov stable. This is true even if there are repeated eigenvalues, as long as the eigenvectors are independent.

If on the other hand there were repeated eigenvalues (say,  $\lambda_j = \lambda_{j+1}$ ) with *dependent* eigenvectors, the general solution would become

$$Q = A_1 \phi^1 e^{\lambda_1 t} + \dots + A_j \phi^j e^{\lambda_j t} + A_{j+1} \phi^j t e^{\lambda_j t} + \dots + A_{12n} \phi^{12n} e^{\lambda_{12n} t} \quad (156)$$

If the real part of  $\lambda_j$  is zero, the function  $t e^{\lambda_j t}$  is unbounded, and the null solution of Eq. (145) is unstable. Liapunov stability is possible in the presence of dependent eigenvectors only if the real part of the corresponding eigenvalues is negative.

Now it will be argued that the null solution of Eq. (143), restricted to constant  $M'$ ,  $G'$ ,  $K'$ , and to zero values of  $A'$  and  $D'$  (and hence of the corresponding Eq. 145) is Liapunov stable, and that the eigenvalues  $\lambda_1, \dots, \lambda_{12n}$  have zero real parts. These conditions have been shown to be incompatible with the presence of dependent eigenvectors.

The function  $\dot{q}^T M' \dot{q} + q^T K' q$  is a Liapunov function (see Ref. 34, Ch. 1) when  $M'$  and  $K'$  are positive definite, so under these conditions the null solution of Eq. (143) (and hence of Eq. 145) is Liapunov stable. By definition (Eq. 87),  $M'$  is positive definite. The matrix  $K'$ , defined by Eq. (91), consists, when  $\omega$  is constant, of the positive-definite stiffness matrix  $K$  plus a matrix provided by the centrifugal "force" field induced by vehicle rotation. Unless centrifugal forces are so great as to exceed the elastic forces of the structure (in which case the appendage would pull apart), the matrix  $K'$  is positive definite. Thus under all reasonable circumstances the null solutions of Eqs. (143) and (145) are Liapunov stable, for the noted restrictions.

Because the matrix  $G'$  in Eq. (143) is skew-symmetric, and the symmetric matrix  $D'$  has been deleted, there can exist no eigenvalues with negative parts. This is a consequence mathematically of the absence of odd-powered terms in the characteristic equation (151). In physical terms, this is a reflection of the fact that the appendage as presently idealized has no damping, and consequently no attenuating solutions are admissible. Thus Liapunov stability of the null solution of Eq. (143) must mean that all eigenvalues  $\lambda_1, \dots, \lambda_{12n}$  have zero real parts, under present restrictive assumptions.

Consequently it can be concluded that, for this class of system, the eigenvectors  $\phi^1, \dots, \phi^{12n}$  are independent, and the matrix  $\Phi^{-1}$  always exists. This means that the operation of Eq. (154) is always possible, and matrix  $B$  can always be diagonalized for the problem under consideration.

In application to the differential equations of the space vehicle, the appropriate transformation procedure is straightforward. Each of the equations involving  $q$  (Eqs. 140, 141, and 134, for example) must be rewritten in terms of  $Q$  as defined in Eq. (144). Thus Eq. (140) becomes

$$\dot{Q} = BQ + L \quad (157)$$

where, in terms of  $6n$  by  $1$  submatrices  $0$  and  $L'$  (see Eq. 140), the  $12n$  by  $1$  matrix  $L$  is  $[0 \ L']^T$ . Equation (141) may be written as

$$I\ddot{\omega} + \tilde{m}I\dot{\omega} + (\tilde{I} + G - \tilde{h})\omega = [0] - (\Sigma_{nk}^T + \tilde{R}\Sigma_{kn}^T + \Sigma_{kn}^T \tilde{r})M] \dot{Q} - \dot{h} - mb\dot{\xi}E^2 + T \quad (158)$$

where the  $3$  by  $12n$  coefficient matrix of  $\dot{Q}$  is shown in two  $3$  by  $6n$  partitions. Recall that in writing Eq. (141) from Eq. (129), the matrices  $I$  and  $G$  were noted to depend on  $q$  and  $\dot{q}$ , so these matrices in Eq. (158) require reformulation in terms of  $Q$ . Similarly, the damper equation (134) and any others involving  $q$  are easily rewritten in terms of  $Q$ . In practice, one would normally carry the transformation into first-order equations all the way, replacing, for example, the second-order scalar damper equation by two first-order equations by introducing a  $6$  by  $1$  matrix for  $[\xi \ \dot{\xi}]^T$  as was done in Eq. (144), and similarly replacing  $\dot{\xi}$  in Eq. (158).

The transformation

$$Q = \Phi Y \quad (159)$$

is next introduced wherever  $Q$  appears in any of the equations of the vehicle or control system, and Eq. (157)

is then premultiplied by  $\Phi^{-1}$ . The transformed appendage equation then becomes (by virtue of Eq. 154)

$$\dot{Y} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{12n} \end{bmatrix} Y + \Phi^{-1} L \quad (160)$$

and the vehicle rotational equation (158) becomes

$$I\dot{\omega} + \tilde{\omega}I\omega + (\dot{I} + G - \tilde{h})\omega = [0] - (\Sigma_{0E}^T + R\tilde{\Sigma}_{0E}^T + \Sigma_{0E}^T \tilde{r})M] \Phi \dot{Y} - \dot{h} - mb\ddot{x}_E + T \quad (161)$$

The scalar equations in Eq. (160) are not directly coupled by the deformation coordinates in  $Y$ . It should be noted, however, that the matrix  $L$  in Eq. (160) is determined by the base rotational motion obtained from Eq. (161), and in general, all of the scalar variables in  $Y$  participate in Eq. (161). Thus when the total system of differential equations is considered, there is still coupling among the deformation coordinates in  $Y$ . Truncation of the  $12n$  by  $1$  matrix  $Y$  and the corresponding  $12n$  by  $12n$  transformation matrix  $\Phi$  must therefore be based on the considered judgment that the coordinate coupling is weak and physically indirect. Such arguments support the practice of determining coordinate truncation principally from the uncoupled appendage deformation equations in Eq. (160). It may be necessary to consider also certain characteristics of the control system, such as the range of frequencies of response, sensor data frequency filtering, and the location of sensors and thrusters. Discussion of these questions of control system interaction is postponed until Section V.

The homogeneous solution for  $Q$  in Eq. (155) is, of course, the same as that obtained by solving Eq. (160) for  $Y$  when  $L$  is zero and substituting into the transformation of Eq. (159). Truncation of  $Y$  is normally accomplished by inspection of this homogeneous solution, although it may also be necessary to include any coordinates in  $Y$  that may introduce resonances due to coincidence of the imaginary part of the corresponding  $\lambda$  with a driving frequency in the forcing term  $L$  in Eq. (160). Since truncation is determined from physical considerations, it is important that the physical significance of the variables  $Y_1, \dots, Y_{12n}$  be understood. This may seem difficult, since complex eigenvectors in  $\Phi$  make the coordinates  $Y_1, \dots, Y_{12n}$  generally complex, even when  $Q$  is by definition real (see Eq. 159). The indicated homogeneous solution of Eq. (160) has the elements

$$Y_m = A_m e^{\lambda_m t} \quad (162)$$

Thus the solution shown in Eq. (155) may be written as

$$Q = \sum_{m=1}^{12n} Y_m \Phi^m \quad (163)$$

in conformance with Eq. (159). Although individual scalars  $Y_m$  and column matrices  $\Phi^m$  are complex, the reality of  $Q$  is assured by the appearance of eigenvalues and eigenvectors in complex pairs. Consider, for example, a solution for  $Y$  with initial conditions selected so that all values of  $A_m$  in Eq. (162) are zero except  $A_K$  and  $A_{nn+K}$ . Then  $Q$  is

$$\begin{aligned} Q &= Y_K \Phi^K + Y_{nn+K} \Phi^{nn+K} \\ &= A_K \Phi^K e^{\lambda_K t} + A_{nn+K} \Phi^{nn+K} e^{\lambda_{nn+K} t} \\ &= A_K \Phi^K e^{\lambda_K t} + A_{nn+K} \Phi^{K*} e^{\lambda_K^* t} \end{aligned}$$

Let  $\lambda_K = \alpha_K + i\sigma_K$  and  $\Phi^K = \Psi^K + i\Gamma^K$  and substitute  $e^{\lambda_K t} = e^{\alpha_K t} (\cos \sigma_K t + i \sin \sigma_K t)$ , etc. Then  $Q$  becomes

$$\begin{aligned} Q &= e^{\alpha_K t} \{ (A_K + A_{nn+K}) (\Psi^K \cos \sigma_K t - \Gamma^K \sin \sigma_K t) \\ &\quad + i (A_K - A_{nn+K}) (\Gamma^K \cos \sigma_K t + \Psi^K \sin \sigma_K t) \} \end{aligned}$$

Since  $Q$  is real,  $A_{nn+K}$  is  $A_K^*$ . Let  $A_K = (C_K - iD_K)/2$ , so that  $Q$  may be rewritten in wholly real terms as

$$\begin{aligned} Q &= e^{\alpha_K t} \{ C_K (\Psi^K \cos \sigma_K t - \Gamma^K \sin \sigma_K t) \\ &\quad + D_K (\Psi^K \sin \sigma_K t + \Gamma^K \cos \sigma_K t) \} \quad (164) \end{aligned}$$

Thus the general homogeneous solution for  $Q$  appears in real terms as

$$\begin{aligned} Q &= \sum_{m=1}^{12n} e^{\alpha_m t} \{ C_m (\Psi^m \cos \sigma_m t - \Gamma^m \sin \sigma_m t) \\ &\quad + D_m (\Psi^m \sin \sigma_m t + \Gamma^m \cos \sigma_m t) \} \quad (165) \end{aligned}$$

The  $12n$  scalars  $C_1, \dots, C_{nn}$  and  $D_1, \dots, D_{nn}$  are established by the initial conditions on  $Q$ . For an undamped system, as in Eq. (143) when  $D' = 0$ , the scalars  $\alpha_1, \dots, \alpha_{nn}$  are all zero.

Now coordinate truncation can be based on the same rationale traditionally employed by structural dynamicists in application to second-order equations of motion. Each conjugate pair of coordinates (say,  $Y_m$  and  $Y_{nn+m} = Y_m^*$ ) corresponds to appendage vibrations of the entire structure at a given frequency (here  $\sigma_m$ ). In general, these vibrations are not in phase throughout the structure, as would be the case for classic normal modes of vibration about a rest configuration. The classic interpretation of the eigenvector  $\Phi^m$  of Eq. (148) as the mode shape of the

vibration at frequency  $\sigma_m$  is modified by the complex nature of  $\phi^m = \psi^m + i\Gamma^m$ . As illustrated in real terms in Eq. (164), this is reflected as two different mode shapes, corresponding to two vibrations at frequency  $\sigma_m$ , with a 90-deg phase lag. Furthermore, these mode shapes depend on the initial conditions. The coordinates in  $Y$  are still called modal coordinates in this report, despite these deviations from the classic interpretation.

The usual decision of the structural dynamicist is to retain in truncation the responses at the lowest natural frequencies, and also any others at which expected driving frequencies are close to natural frequencies (with consequent danger of resonance). The justification for concentrating on lower frequency responses is often based on the fact that storing a given amount of energy as potential energy or strain energy in the deformation of an elastic structure into a low-frequency mode shape generally results in larger displacements and base reactions than are present when the same energy is stored in a high-frequency mode shape deformation. To this it may be added that the present elastic model is unrealistic in its neglect of damping, and that, in fact, the higher frequency vibrations will have more damping than those at low frequency. This again supports the practice of ignoring the high-frequency response (since it rapidly attenuates in the transient solution anyway).

The appendage deformation matrix  $Y$  in Eqs. (160) and (161) is therefore replaced by the  $2N$  by 1 truncated matrix

$$\bar{Y} = [Y_1 \cdots Y_N Y_1^* \cdots Y_N^*]^T \quad (166)$$

where  $N$  is the number of modes to be preserved in the simulation. The transformation matrix  $\Phi$  is accordingly truncated to the  $12n$  by  $2N$  matrix  $\bar{\Phi}$ , where

$$\bar{\Phi} = [\phi^1 \cdots \phi^N \phi^{1*} \cdots \phi^{N*}] \quad (167)$$

The equation of motion of the appendage (Eq. 160) now becomes

$$\dot{\bar{Y}} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \ddots & & \\ & & \lambda_N & \\ & & & \lambda_1^* \\ & & & & \ddots \\ 0 & & & & & \lambda_N^* \end{bmatrix} \bar{Y} + \bar{\Phi}^{-1} L \quad (168)$$

This matrix equation consists of  $2N$  scalar equations in the complex variables  $Y_1, \dots, Y_N, Y_1^*, \dots, Y_N^*$ . In com-

putation, only the first  $N$  equations need be considered, since the equations are uncoupled and the second set of  $N$  variables is available as the complex conjugates of the first set.

The rotational equation of the vehicle (Eq. 161) then becomes

$$I\dot{\omega} + \tilde{\omega}I\omega + (\dot{I} + G - \tilde{h})\omega = [0] - (\Sigma_{\omega E}^T + \tilde{R}\Sigma_{E\omega}^T + \Sigma_{E\omega}^T \tilde{r})M] \bar{\Phi} \dot{\bar{Y}} - \dot{h} - mb\ddot{E}^2 + T \quad (169)$$

The matrix  $\bar{\Phi}$  is rectangular, and the symbol  $\bar{\Phi}^{-1}$  in Eq. (168) represents a left inverse of  $\bar{\Phi}$ . This is available (Ref. 35) as

$$\bar{\Phi}^{-1} = (\bar{\Phi}^T \bar{\Phi})^{-1} \bar{\Phi}^T \quad (170)$$

as may be confirmed by the multiplication

$$[(\bar{\Phi}^T \bar{\Phi})^{-1} \bar{\Phi}^T] \bar{\Phi} = (\bar{\Phi}^T \bar{\Phi})^{-1} (\bar{\Phi}^T \bar{\Phi}) = E$$

The inversion actually performed in Eq. (170) to obtain  $\bar{\Phi}^{-1}$  is of a matrix of dimension  $2N$ . Computations are thus not as imposing a task as may have been implied by the presence of the  $12n$  by  $12n$  matrix inverse  $\Phi^{-1}$  in Eq. (160). In practice, the numerical value of  $\Phi$  is never computed, since only  $\bar{\Phi}$  is required.

Incorporation of energy dissipation into the appendage mathematical model is usually deferred, pending transformation to modal coordinates. The scalar  $\alpha_k$  in Eq. (164) would be zero for the undamped system of Eq. (143) with  $D' = 0$ , since  $G'$  is skew-symmetric. Conventional practice in structural dynamics is to assign values to  $\alpha_1, \dots, \alpha_N$  (where  $N$  modes are retained in the modal-coordinate truncation), with preliminary numerical values based on experience with similar structures and final values based on prototype tests. This amounts to rather arbitrarily adding a viscous damping term  $D' \dot{q}$  to Eq. (143) after completing the modal analysis, and assuming that the structure of the matrix  $D'$  is such that the eigenvectors  $\phi^1, \dots, \phi^{12n}$  are undisturbed by this addition. Numerical values of the elements in  $D'$  are never considered, since they are determined implicitly by the choice of  $\alpha_1, \dots, \alpha_N$ . (Actually, the conventional pattern is the assignment of a value to  $\xi_m = \alpha_m / \sigma_m$  rather than to  $\alpha_m$  itself. Here  $\xi_m$  is called the "percentage of critical damping." For typical spacecraft appendage structures,  $\xi_m$  is well below 5% for the lower frequency responses.)

The mechanical process of incorporating damping in the appendage equations is straightforward. Equation (168) includes a diagonal matrix of eigenvalues  $\lambda_1, \dots, \lambda_N, \lambda_1^*, \dots, \lambda_N^*$ . As calculated from the matrix  $B$  of Eq. (146) with  $D'$  set equal to zero, these eigenvalues will be purely imaginary, e.g.,  $\lambda_m = i\sigma_m$ . To incorporate structural damping, one instead substitutes  $\lambda_m = \xi_m \sigma_m + i\sigma_m$  into Eq. (168), where  $\xi_m$  is chosen as noted in the preceding paragraph.

It should be noted that the procedure just described for transformation to modal coordinates and subsequent truncation did not require any special properties of the constant matrix  $B$  in Eq. (145), except in proving the existence of  $\Phi^{-1}$  in Eq. (154). The nonsingularity of  $\Phi$  was established by relying upon the skew-symmetry of matrix  $G'$  and the absence of the viscous damping matrix  $D'$  in Eq. (143). This property was used to prove that even in the special case of repeated eigenvalues, the independence of the eigenvectors is assured, so the solution to Eq. (145) has the form of Eq. (155) rather than that of Eq. (156). The proof would proceed equally well if  $G'$  were zero, but when damping is present the formal argument underlying the previous discussion breaks down. Although it seems reasonable that continuity would require the preservation of the nonsingularity of  $\Phi$  in the presence of small damping forces (with the corresponding  $D' \dot{q}$  included in Eq. 143, with  $D'$  symmetric), proof of this condition seems lacking.

It can be asserted, then, that only when  $\Phi^{-1}$  exists is the procedure applied here to transform the state equation (145) into the diagonal form applicable. It is perhaps worth repeating that the existence of  $\Phi^{-1}$  is assured when the eigenvalues of  $B$  are distinct, or when the matrix  $D'$  is zero. This method has thus been shown to be applicable to any linear dynamic system with distinct natural frequencies, and to any undamped linear dynamic system. Applicability to damped linear dynamic systems with repeated natural frequencies seems probable, but has not been formally established.

**3. Modal analysis of damped, nonrotating structures.** The utility of the direct analysis of the state equation may depend on the computational efficiency of the inversion of  $\Phi$  (after truncation). It should be noted that  $\Phi$  is not orthogonal, i.e.,  $\Phi^{-1} \neq \Phi^T$ . Foss (Ref. 36) has shown that when  $G'$  is zero,  $D'$  is nonzero, and eigenvalues are distinct, certain weighted orthogonality relationships can be established. Foss notes the equivalence of Eqs. (157) and (140) with the first-order equation

$$M'' \dot{Q} + K'' Q = L'' \quad (171)$$

where

$$M'' = \begin{bmatrix} D' & M' \\ M' & 0 \end{bmatrix}$$

$$K'' = \begin{bmatrix} K' & 0 \\ 0 & M' \end{bmatrix}$$

$$L'' = \begin{bmatrix} L' \\ 0 \end{bmatrix}$$

Foss demonstrates that, for symmetric  $D'$ , the eigenvectors  $\phi^m$  and  $\phi^k$  corresponding to different eigenvalues ( $\lambda_m \neq \lambda_k$ ) have the orthogonality property

$$\left. \begin{aligned} \phi^{mT} M'' \phi^k &= 0 \\ \phi^{mT} K'' \phi^k &= 0 \end{aligned} \right\} \quad (172)$$

Thus for suitable eigenvector normalization, diagonalization of Eq. (171) with the transformation (see Eq. 159)

$$\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = Q = \Phi Y$$

is computationally easier than the diagonalization of Eq. (145). Whereas the latter procedure requires the calculation of  $\Phi^{-1}$  (or  $\bar{\Phi}^{-1}$  after truncation), the former requires only the construction of  $\Phi^T$  (or  $\bar{\Phi}^T$ ) to obtain the same result. This transformation of Eq. (171), followed by premultiplication by  $\Phi^T$ , provides the same results as Eq. (168), namely,

$$\Phi^T M'' \Phi \dot{Y} + \Phi^T K'' \Phi Y = \Phi^T L''$$

or, with Eq. (172) and truncation,

$$\dot{\bar{Y}} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \ddots & & \\ & & \lambda_N & \\ 0 & & & \lambda_1^* & \ddots & \\ & & & & \ddots & \\ & & & & & \lambda_N^* \end{bmatrix} \bar{Y} = \bar{\Phi}^T L'' \quad (173)$$

This equation combines with the vehicle equation (169) and parallel equations for rotor and damper to obtain a complete simulation. Equation (173) is not applicable to the differential equations of an appendage on a rotating base (Eqs. 140 and 143 with  $G' \neq 0$ ). When the base is not rotating, and specific damping forces are necessary (perhaps because of discrete dashpots in the system), the method of Foss is computationally superior, but mathematically equivalent, to the more general method used to derive Eq. (168). Both of these methods have two major undesirable features:

- (1) Except in the very special case for which  $G' = 0$  and  $D'$  is a polynomial in the symmetric matrices  $M'$  and  $K'$  (Ref. 30), the eigenvectors forming the columns of the transformation matrix  $\Phi$  are complex, and thus the new deformation variables  $Y$  introduced in Eq. (159) are also complex. Although physical interpretation of the complex coordinates (as conjugate pairs) has been provided, this interpretation is less immediate and probably less comfortable for the engineer than would be a system of real coordinates. New computer programs might be necessitated by the simulation in terms of complex coordinates, whereas existing programs might suffice if a useful transformation to real coordinates can be found.
- (2) The final equation of motion of the appendage (Eq. 168 or Eq. 173) is a first-order matrix equation. Although controls engineers are finding increasing use for first-order equations, the structural dynamicist is more accustomed to working with second-order equations. Computation with existing computer programs may be precluded and interpretation of results may be impeded by the adoption of first-order equations.

#### 4. Modal analysis of undamped, rotating structures.

When in Eq. (143) the acceleration matrix  $A'$  and the damping matrix  $D'$  are zero, an alternative to the transformation of Eq. (159) can be devised that results ultimately in a system of uncoupled, real, second-order scalar equations of motion. This method is inapplicable when damping is included in the mathematical model of the appendage structure, although the introduction of modal damping into the transformed equations is not precluded. The transformation to be described will uncouple the homogeneous second-order appendage equations even when the appendage is on a rotating base (so that  $G' \neq 0$  in Eq. 143), but this will not be successful if a discrete dashpot is incorporated into the appendage or the appendage support structure, since then  $D' \neq 0$  in Eq. (143).

The general solution for Eq. (143) with  $A' = 0$  has already been found, since  $q$  is identified in Eq. (144) as the upper half of  $Q$ , and the solution for  $Q$  has been recorded in real terms in Eq. (165). As noted previously,  $\alpha_m$  is zero for all  $m$  for undamped systems. If now the upper halves of  $\Psi$  and  $\Gamma$  are denoted by the corresponding lower-case letters (as in Eq. 148 for  $\Phi$ ), the substitutions

$$\left. \begin{aligned} \Psi^m &= \begin{bmatrix} \psi^m \\ -\psi^m \lambda_m \end{bmatrix} \\ \Gamma^m &= \begin{bmatrix} \gamma^m \\ -\gamma^m \lambda_m \end{bmatrix} \end{aligned} \right\} \quad (174)$$

permit  $q$  to be written from Eq. (165) in the form

$$q = \sum_{m=1}^{2n} \{ C_m (\psi^m \cos \sigma_m t - \gamma^m \sin \sigma_m t) + D_m (\psi^m \sin \sigma_m t + \gamma^m \cos \sigma_m t) \} \quad (175)$$

The derivative  $\dot{q}$  may be obtained by differentiating  $q$  or by inspecting the bottom half of Eq. (165), as

$$\dot{q} = \sum_{m=1}^{2n} \{ -C_m \sigma_m (\psi^m \sin \sigma_m t + \gamma^m \cos \sigma_m t) + D_m \sigma_m (\psi^m \cos \sigma_m t - \gamma^m \sin \sigma_m t) \} \quad (176)$$

If now  $Z^h$  is defined as the  $12n$  by 1 matrix

$$Z^h \equiv \begin{bmatrix} C_1 \cos \sigma_1 t + D_1 \sin \sigma_1 t \\ C_2 \cos \sigma_2 t + D_2 \sin \sigma_2 t \\ \vdots \\ C_{2n} \cos \sigma_{2n} t + D_{2n} \sin \sigma_{2n} t \\ -C_1 \sin \sigma_1 t + D_1 \cos \sigma_1 t \\ \vdots \\ -C_{2n} \sin \sigma_{2n} t + D_{2n} \cos \sigma_{2n} t \end{bmatrix} \quad (177)$$

then by inspection of Eqs. (165) and (177), one may write

$$Q = PZ^h \quad (178)$$

where  $P$  is the transformation matrix

$$P \equiv \begin{bmatrix} \psi^1 & \psi^2 & \cdots & \psi^{6n} & \gamma^1 & \gamma^2 & \cdots & \gamma^{6n} \\ -\sigma_1 \gamma^1 & -\sigma_2 \gamma^2 & \cdots & -\sigma_{6n} \gamma^{6n} & \sigma_1 \psi^1 & \sigma_2 \psi^2 & \cdots & \sigma_{6n} \psi^{6n} \end{bmatrix} \quad (179)$$

By inspection of the matrix  $Z^h$  and its time derivative, one may write

$$\dot{Z}^h = \begin{bmatrix} 0 & \sigma_1 & 0 \\ & \ddots & \\ -\sigma_1 & 0 & \sigma_{6n} \\ 0 & & 0 \end{bmatrix} Z^h \equiv \begin{bmatrix} 0 & \sigma \\ -\sigma & 0 \end{bmatrix} Z^h \quad (180)$$

where  $\sigma$  has been defined as the  $6n$  by  $6n$  diagonal matrix of natural frequencies  $\sigma_1, \dots, \sigma_{6n}$ .

The first  $6n$  elements of  $Z^h$  may be assigned the symbols  $z_1, \dots, z_{6n}$ , so that Eq. (177) may be written as

$$Z^h = \begin{bmatrix} z_1 \\ \vdots \\ z_{6n} \\ \dot{z}_1/\sigma_1 \\ \vdots \\ \dot{z}_{6n}/\sigma_{6n} \end{bmatrix} \equiv \begin{bmatrix} z \\ \sigma^{-1} \dot{z} \end{bmatrix} \quad (181)$$

which defines the  $6n$  by 1 column matrix  $z$  and the  $6n$  by  $6n$  diagonal matrix  $\sigma^{-1}$  whose elements are the frequency reciprocals  $1/\sigma_1, \dots, 1/\sigma_{6n}$ . In terms of  $z_1, \dots, z_{6n}$ , Eq. (180) becomes the set of scalar equations

$$\left. \begin{aligned} \dot{z}_1 &= \dot{z}_1 \\ &\vdots \\ \dot{z}_{6n} &= \dot{z}_{6n} \\ \ddot{z}_1 &= -\sigma_1^2 z_1 \\ &\vdots \\ \ddot{z}_{6n} &= -\sigma_{6n}^2 z_{6n} \end{aligned} \right\} \quad (182)$$

The first set of  $6n$  scalar equations in Eq. (182) is trivial, but the second set is a group of  $6n$  uncoupled equations

in the real variables  $z_1, \dots, z_{6n}$ , which is equivalent to Eq. (143). In matrix terms, this result is

$$\ddot{z} + \sigma^2 z = 0 \quad (183)$$

where

$$\sigma^2 \equiv \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_{6n}^2 \end{bmatrix}$$

To accomplish a direct transformation from  $q$  in Eq. (143) to  $z$  in Eq. (183), rewrite the transformation in Eq. (144) as

$$q \equiv [E \mid 0] Q \quad (184)$$

where the identity matrix  $E$  and the null matrix  $0$  are of dimension  $6n$  by  $6n$ . Substitution of Eqs. (178) and (181) into Eq. (184) produces

$$q = [E \mid 0] P \begin{bmatrix} z \\ \sigma^{-1} \dot{z} \end{bmatrix} \quad (185)$$

or, from Eq. (179),

$$q = \psi z + \gamma \sigma^{-1} \dot{z} \quad (186)$$

where  $\psi$  and  $\gamma$  are the  $6n$  by  $6n$  matrices

$$\psi \equiv [\psi^1 \psi^2 \cdots \psi^{6n}]$$

and

$$\gamma \equiv [\gamma^1 \gamma^2 \cdots \gamma^{6n}]$$

A transformation of the character of Eq. (185), which relates old coordinates to a combination of new coordinates and their derivatives (or, more often, their generalized momenta), is known as a *contact transformation*. In contrast, a transformation that relates one set of generalized coordinates directly to another is called a *point transformation*. As noted in Whittaker (Ref. 24, p. 195, footnote), there exists no *point* transformation that takes equations of the structure of Eq. (143) with  $A' \equiv D' \equiv 0$



into uncoupled second-order equations such as Eq. (183), unless  $G'$  is also zero. The existence of a *contact* transformation that accomplishes this objective when  $G' \neq 0$  is noted in Whittaker (p. 427), where equations of motion are expressed in terms of Hamilton's canonical equations.

Although Eq. (143) has been the starting point in the quest for useful transformations, it is the inhomogeneous counterpart, Eq. (140), that must actually be employed in the simulation. In terms of first-order equations, it is Eq. (157) and not Eq. (145) that must be used. Direct substitution of the transformation in Eq. (186) into the second-order inhomogeneous Eq. (140) does not appear to be fruitful. Instead the transformation

$$Q = PZ \quad (187)$$

is used in the first-order inhomogeneous Eq. (157), where  $P$  is a matrix of constants available from the eigenvalues and eigenvectors of  $B$  as in Eq. (179), and  $Z$  is a new matrix of unknowns. The result is, from Eqs. (157) and (187),

$$P\dot{Z} = BPZ + L \quad (188)$$

Only when  $L$  is zero is the solution for  $Z$  that given as  $Z^h$  by Eq. (177).

When  $P$  is nonsingular (a condition not always assured), Eq. (188) may be written as

$$\dot{Z} = P^{-1}BPZ + P^{-1}L \quad (189)$$

The matrix  $P^{-1}BP$  has been shown in the context of the analysis of the homogeneous equation (see Eq. 180) to have the value

$$P^{-1}BP = \left[ \begin{array}{c|c} 0 & \sigma \\ \hline -\sigma & 0 \end{array} \right] \quad (190)$$

and this product of constant matrices cannot change its value when the equation becomes inhomogeneous.

The inverse of  $P$  as required in Eq. (189) is expressed and computed most easily in terms of its partitioned elements, according to the formula (Ref. 37, p. 640)

$$P^{-1} = \left[ \begin{array}{c|c} (P_{11} - P_{12}P_{22}^{-1}P_{21})^{-1} & -P_{11}^{-1}P_{12}(P_{22} - P_{21}P_{11}^{-1}P_{12})^{-1} \\ \hline -P_{22}^{-1}P_{21}(P_{11} - P_{12}P_{22}^{-1}P_{21})^{-1} & (P_{22} - P_{21}P_{11}^{-1}P_{12})^{-1} \end{array} \right] \quad (191)$$

where

$$P = \left[ \begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array} \right] = \left[ \begin{array}{c|c} \psi & \gamma \\ \hline -\sigma\gamma & \sigma\psi \end{array} \right] \quad (192)$$

With the identity typified by

$$(\sigma\psi)^{-1} = \psi^{-1}\sigma^{-1} \quad (193)$$

the inverse of  $P$  simplifies to

$$P^{-1} = \left[ \begin{array}{c|c} (\psi + \gamma\psi^{-1}\gamma)^{-1} & -\psi^{-1}\gamma(\psi + \gamma\psi^{-1}\gamma)^{-1}\sigma^{-1} \\ \hline \psi^{-1}\gamma(\psi + \gamma\psi^{-1}\gamma)^{-1} & (\psi + \gamma\psi^{-1}\gamma)^{-1}\sigma^{-1} \end{array} \right] \quad (194)$$

In expanded form, Eq. (189) may now be written (see Eq. 157 for  $L$  expansion) in the form

$$\begin{bmatrix} \dot{Z}_1 \\ \vdots \\ \dot{Z}_{12n} \end{bmatrix} = \begin{bmatrix} 0 & \sigma_1 & 0 \\ & \ddots & \\ -\sigma_1 & 0 & \sigma_{6n} \\ & \ddots & \\ 0 & -\sigma_{6n} & 0 \end{bmatrix} \begin{bmatrix} Z_1 \\ \vdots \\ Z_{12n} \end{bmatrix} + \begin{bmatrix} (\psi + \gamma\psi^{-1}\gamma)^{-1} & -\psi^{-1}\gamma(\psi + \gamma\psi^{-1}\gamma)^{-1}\sigma^{-1} \\ \psi^{-1}\gamma(\psi + \gamma\psi^{-1}\gamma)^{-1} & (\psi + \gamma\psi^{-1}\gamma)^{-1}\sigma^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ L' \end{bmatrix} \quad (195)$$

In the analysis of the homogeneous equation, with  $L'$  absent, the elements of the column matrix  $Z$ , labeled  $Z^h$ , can be identified as in Eq. (181). This permits the scalar equations obtained from the upper half of  $Z$  to be discarded as trivial identities (see Eq. 182), and leaves the lower half of  $Z$  to provide nontrivial, uncoupled second-order equations in the new variables  $z_1, \dots, z_{6n}$  (Eq. 183). This identification and resolution into second-order equations does not appear to be possible in the inhomogeneous case (Eq. 195), unless  $-\psi^{-1}\gamma(\psi + \gamma\psi^{-1}\gamma)^{-1}\sigma^{-1}$  is zero. Note that the inverted matrix in parentheses cannot be zero, or every term in  $P^{-1}$  is zero, and the applied forces and torques in  $L'$  have no dynamic consequences. The most useful condition under which Eq. (195) permits this identification of the elements of  $Z$  as in Eq. (181) is the condition  $\gamma = 0$ . This corresponds to the special case of real eigenvectors, which implies the absence of  $G'$  in Eqs. (143) and (140). This is then the classic problem of the vibration of a structure about a state of inertial rest, to which attention will return shortly.

In general, the  $12n$  first-order equations in Eq. (195) do not admit of reduction to second-order equations in  $z$  as in the homogeneous case (Eq. 183). This reduction does serve, nonetheless, to motivate the coordinate truncation of the deformation variable matrix  $Z$  in the inhomogeneous Eq. (195).

With the use of frequency criteria, the truncation of the  $6n$  by  $1$  matrix  $z$  in Eq. (183) is completely straightforward; one simply ignores all but  $z_1, \dots, z_N$  of the elements in  $z$  when frequencies  $\sigma_1, \dots, \sigma_N$  are of interest. The coordinates  $z_1, \dots, z_N$  are modal coordinates in the broad sense in which that term is used in this report. If the solutions to the scalar equations in Eq. (183) are written as

$$\left. \begin{aligned} z_m &= C_m \cos \sigma_m t + D_m \sin \sigma_m t \\ \dot{z}_m &= -\sigma_m^{-1} C_m \sin \sigma_m t + \sigma_m^{-1} D_m \cos \sigma_m t \end{aligned} \right\} \quad (196)$$

then substitution into Eq. (186) yields the solution for  $q$  in Eq. (175), for which physical interpretation has been provided.

The introduction of energy-dissipation capability into the mathematical model can also be accomplished in the homogeneous case after transformation to the modal-coordinate matrix  $z$ . Equation (183) is simply replaced by

$$\ddot{\bar{z}} + 2\bar{\zeta}\dot{\bar{z}} + \bar{\sigma}^2\bar{z} = 0 \quad (197)$$

where the overbars indicate truncation to  $N$  coordinates, and

$$\bar{\zeta} = \begin{bmatrix} \zeta_1 & 0 \\ & \ddots \\ 0 & \zeta_N \end{bmatrix}$$

In the inhomogeneous case, which is of primary interest, the matrix  $Z$  in Eq. (195) can be written in partitioned terms as

$$Z = \begin{bmatrix} Z^1 \\ Z^2 \end{bmatrix} \quad (198)$$

where  $Z^1$  and  $Z^2$  are  $6n$  by  $1$  matrices. The transformation derived as Eq. (186) for the homogeneous case then becomes

$$q = \psi Z^1 + \gamma Z^2 \quad (199)$$

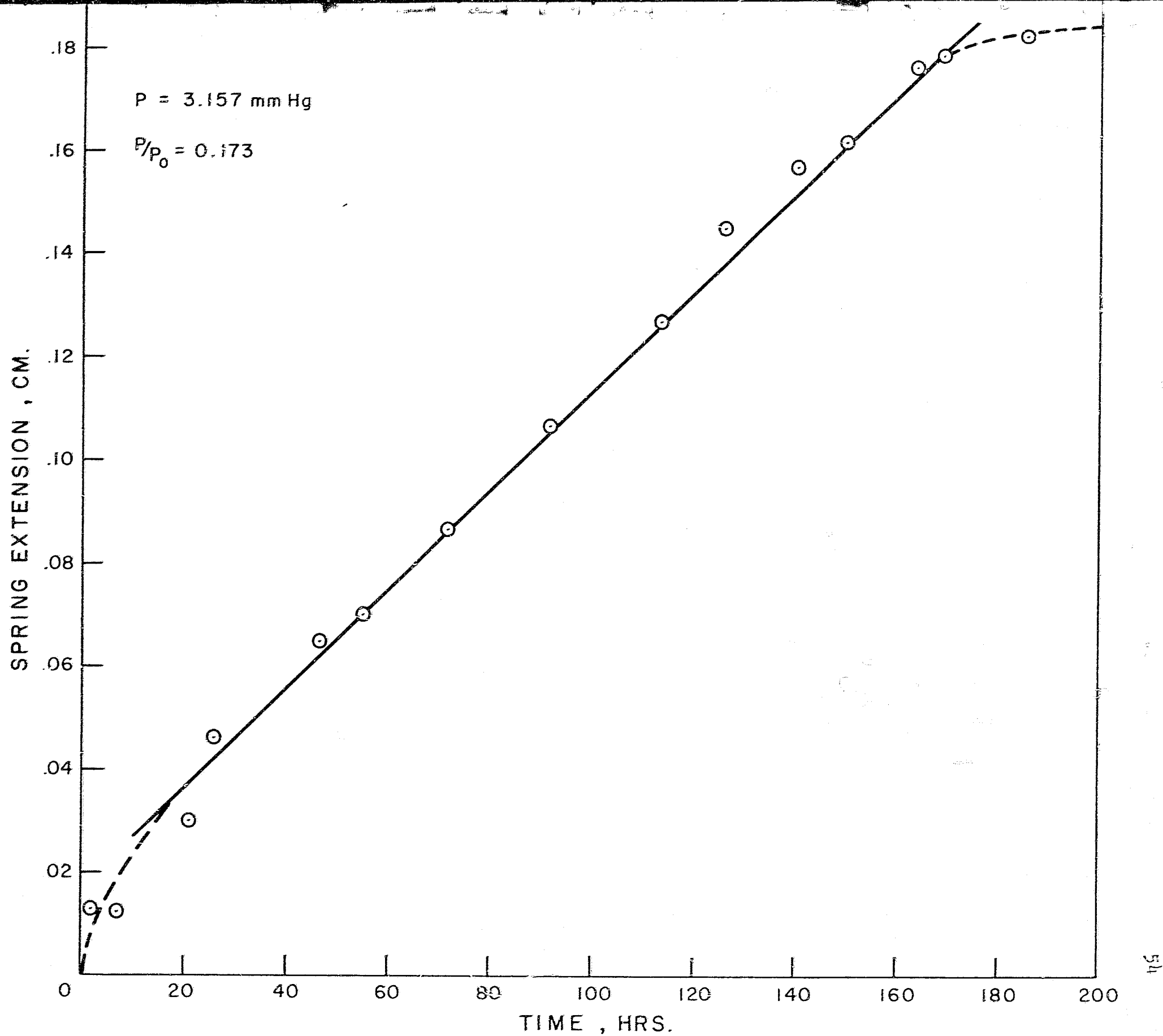


FIG. 8 RATE OF ADSORPTION, WATER - GRAPHON SYSTEM

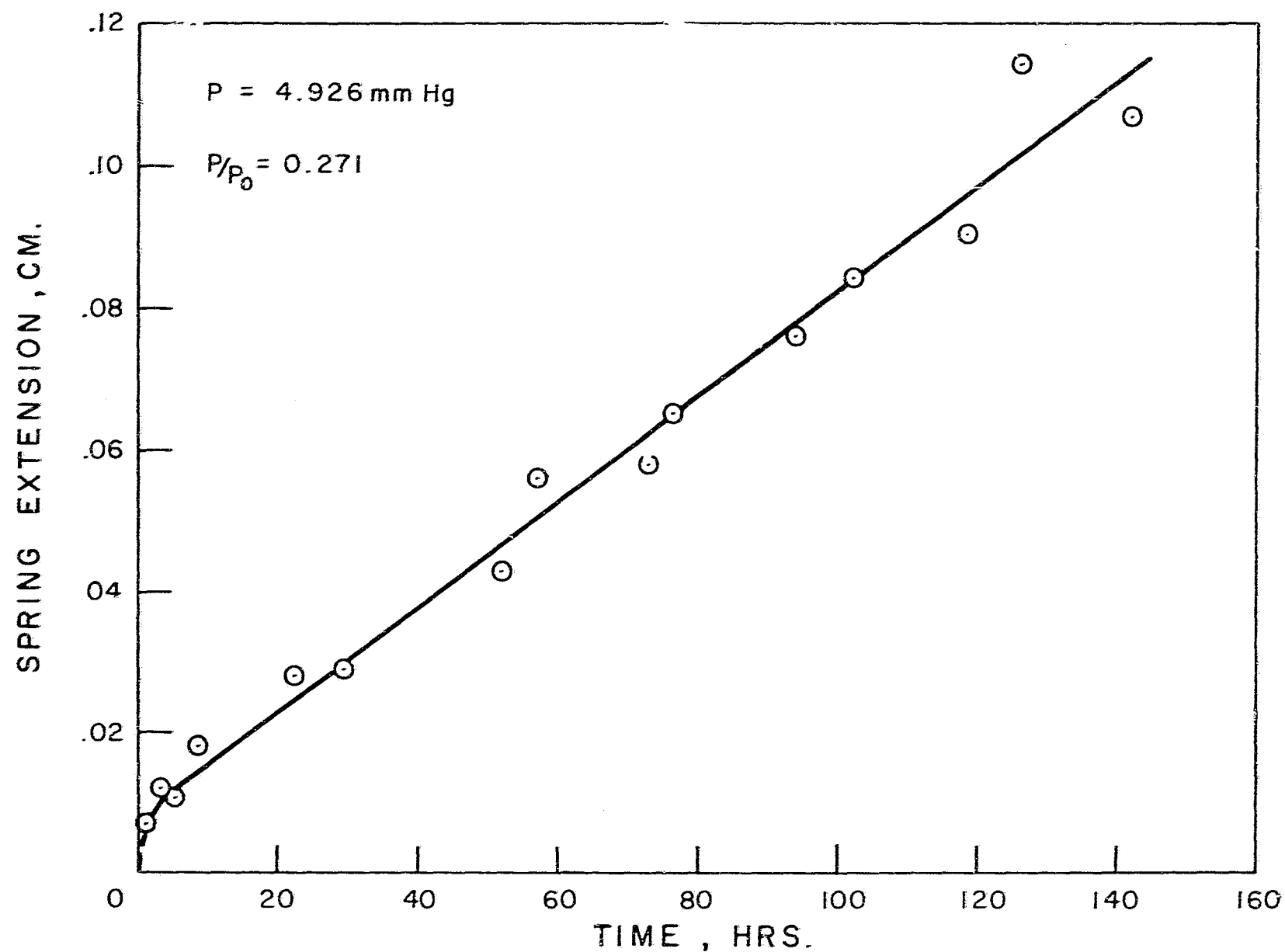


FIG. 9 RATE OF ADSORPTION, WATER - GRAPHON SYSTEM

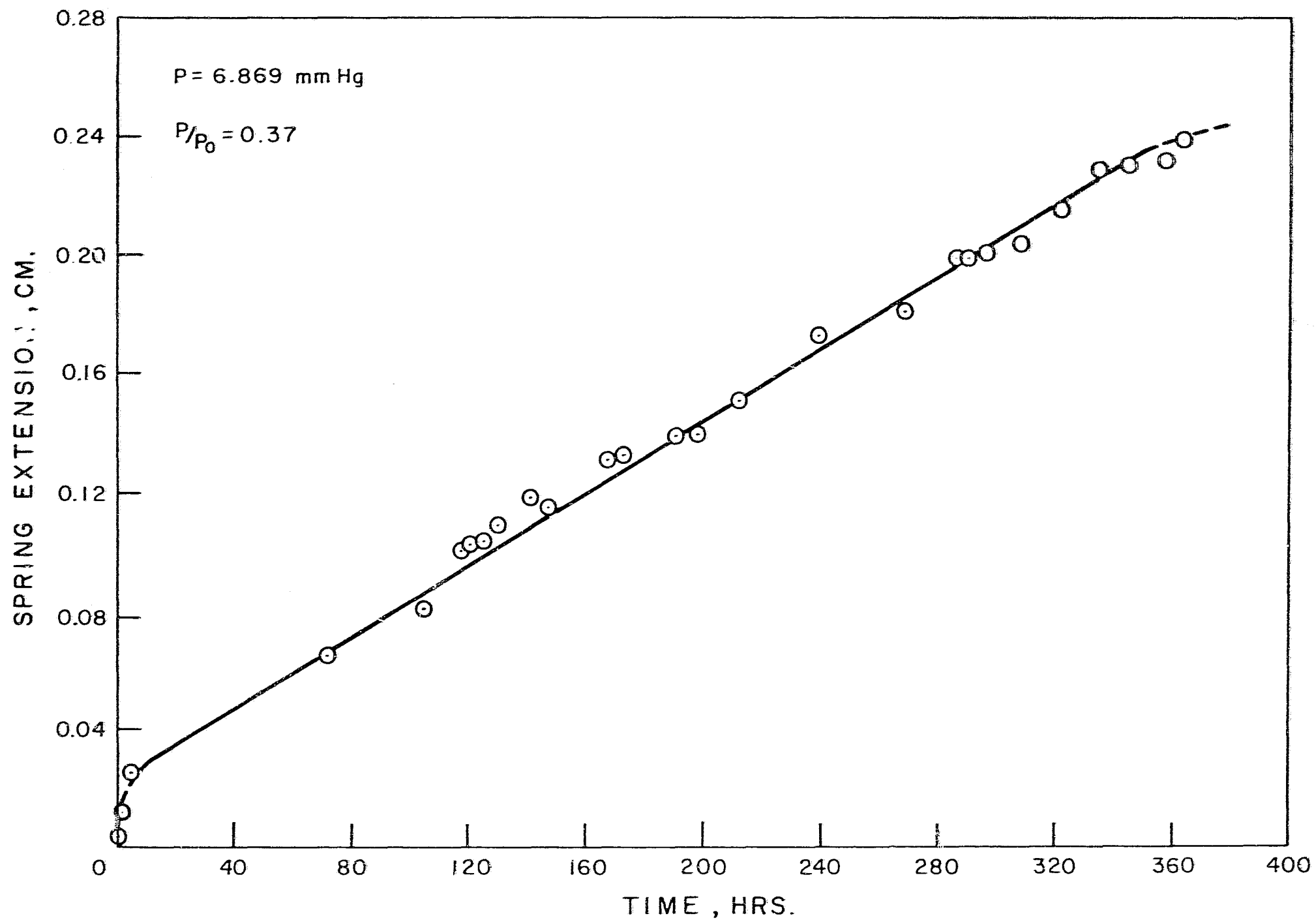


FIG. 10 RATE OF ADSORPTION, WATER - GRAPHON SYSTEM